

Math251H quiz IV - answer

20 minutes

Name:

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}, \quad \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$
$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}, \quad \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s), \quad \mathcal{L}\{e^{ct}f(t)\} = F(s-c)$$

1. (20 points) Use **Laplace transform** to solve the initial value problem:

$$y'' + 2y' - 8y = 4e^{-3t}, \quad y(0) = 4, y'(0) = 1.$$

Solution:

Apply Laplace transform to $y(t)$ and its derivatives,

$$\begin{aligned}\mathcal{L}\{y\} &= Y \\ \mathcal{L}\{y'\} &= sY - 4 \\ \mathcal{L}\{y''\} &= s^2Y - 4s - 1.\end{aligned}$$

With a direct substitution, we get,

$$(s^2 + 2s - 8)Y = 4s + 9 + \frac{4}{s+3} = \frac{4s^2 + 21s + 31}{s+3}.$$

It follows that,

$$Y = \frac{4s^2 + 21s + 31}{(s+4)(s-2)(s+3)} = \frac{11}{6} \frac{1}{s+4} + \frac{89}{30} \frac{1}{s-2} - \frac{4}{5} \frac{1}{s+3}.$$

The last step was done by a partial fraction.

This yields,

$$y(t) = \frac{11}{6}e^{-4t} + \frac{89}{30}e^{2t} - \frac{4}{5}e^{-3t}.$$

2. (20 points) Find the inverse Laplace transform of the following function,

$$F(s) = e^{-4s} \frac{s + 5}{s^2 + 4s + 8}.$$

Solution:

Let $f(t)$ be the inverse Laplace transform of $F(s)$.

1. If we let,

$$G(s) = \frac{s + 5}{s^2 + 4s + 8},$$

and $g(t)$ be inverse Laplace transform. Then, we have,

$$F(s) = G(s)e^{-4s}$$

and therefore

$$f(t) = u_4(t)g(t - 4).$$

2. Meanwhile, if we let,

$$H(s) = \frac{s + 3}{s^2 + 2^2},$$

and $h(t)$ be the inverse Laplace transform. Then we have,

$$G(s) = H(s + 2).$$

As a result,

$$g(t) = h(t)e^{-2t}.$$

3. Since,

$$H(s) = \frac{s}{s^2 + 4} + \frac{3}{2} \frac{2}{s^2 + 4},$$

we have,

$$h(t) = \cos 2t + \frac{3}{2} \sin 2t.$$

Collecting terms, we get the solution,

$$f(t) = u_4(t)e^{-2t+8} \left[\cos(2t - 8) + \frac{3}{2} \sin(2t - 8) \right].$$