MATH251H Practice Exam I

Spring 2008

120 minutes

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1. (15 points) Consider the linear system,

$$\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ -8 & 0 \end{pmatrix} \mathbf{x}.$$

a. Find the general real-valued solutions.

b. Classify the type and stability of the critical point (0,0).

2. (15 points) Consider the linear system,

$$\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} \mathbf{x}.$$

If

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ a \end{pmatrix},$$

and,

$$\lim_{t \to +\infty} \mathbf{x}(t) = \mathbf{0},$$

what is the value of a?

3. (20 points) Consider the following nonlinear system:

$$\begin{cases} x' &= x^2 + y^2 - 10 \\ y' &= 2x - 6y \end{cases}$$

a. Find all the critical points.

b. For each critical point, linearize the system. What conclusions can you draw about the type and stability of the critical points of the nonlinear system?

4. (10 points) Use the method of separation of variables to reduce the following partial differential equation to two ordinary differential equations,

$$u_{xx} - u_{tx} + 5x^3 u_t = 0.$$

Do not solve the ordinary differential equations.

5. (15 points) Solve the heat conduction equation,

$$\begin{cases} u_t &= 4u_{xx}, \quad 0 \le x \le 3, \\ u(0,t) &= 0 \\ u(3,t) &= 0 \\ u(x,0) &= \sin(\frac{2\pi}{3}x) - \sin(\pi x) + 7\sin(\frac{5\pi}{3}x) \end{cases}$$

6. (10 points) Solve the eigenvalue problem,

$$X'' = \lambda X, 0 \leq x \leq \pi,$$

with boundary conditions,

$$X'(0) = 0, X'(\pi) = 0.$$

Find all eigenvalues and eigenfunctions.

7. (15 points) Solve the wave equation,

$$\begin{cases} u_{tt} &= 9u_{xx}, \quad 0 \le x \le 4, \\ u(0,t) &= 0 \\ u(3,t) &= 0 \\ u(x,0) &= 0 \\ u_t(x,t) &= g(x), \end{cases}$$

where $g(x)$ is given by,

$$g(x) = \begin{cases} 1, & 2 \le x \le 3, \\ 0, & \text{otherwise.} \end{cases}$$