THE REGULARIZING EFFECTS OF resetting in a PARTICLE SYSTEM FOR THE BURGERS’ EQUATION

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Abstract. We study the dissipation mechanism of a stochastic particle system for the Burgers’ equation. The velocity field of the viscous Burgers’ and Navier-Stokes equations can be expressed as an expected value of a stochastic process based on noisy particle trajectories (Constantin, Iyer, Comm. Pure Appl. Math, 2008). In this paper we study a particle system for the viscous Burgers’ equations using a Monte-Carlo version of the above; we consider $N$ copies of the above stochastic flow, each driven by independent Wiener processes, and replace the expected value with $\frac{1}{N}$ times the sum over these copies. A similar construction for the Navier-Stokes equations was studied by J. Mattingly and the first author (arXiv:0803.1222, to appear in Nonlinearity).

Surprisingly, for any finite $N$, the particle system for the Burgers’ equations shocks almost surely in finite time. In contrast to the full expected value, the empirical mean $\frac{1}{N} \sum_1^N$ does not regularize the system enough to ensure a time global solution. To avoid these shocks, we consider a resetting procedure, which at first sight should have no regularizing effect at all. We however prove that this procedure prevents the formation of shocks for any $N \geq 2$, and consequently as $N \to \infty$ we get convergence to the solution of the viscous Burgers’ equation on long time intervals.

1. Introduction

The viscous Burgers’ equation

$$\partial_t u + u \partial_x u - \nu \partial_x^2 u = 0$$

(1.1)

has been studied extensively from several different points of view. Here $\nu > 0$ represents the viscosity, making the equation dissipative in nature. The inviscid Burgers’ equation, (equation (1.1) with $\nu = 0$) is studied as the basic example of a scalar conservation law (see e.g. [5,7]). The Burgers’ equation is also linked to the KAM and Aubry-Mather theories [8,12]. It is the simplest PDE that models the Euler and the Navier-Stokes nonlinearity. As such, it has been extensively studied as the first step in understanding the two key unresolved issues in fluid mechanics: turbulence and regularity of the Navier-Stokes equations in three dimensions. In the first category the objective is to characterize the statistical properties of turbulence [6]. In the second category the objective is to understand the regularizing mechanism of dissipation [1,13]. This paper falls into the latter category: we study the regularising mechanism of a particle system for the Burgers’ equations, analogous to the particle system for the Navier-Stokes equations developed in [4,11].
In [4], a class of second order non-linear transport equations (the Navier-Stokes and viscous Burgers’ in particular) were formulated as the average of a stochastic process along noisy particle trajectories. The formulation for the Navier-Stokes equations developed in [4] involves recovering the velocity \( u \) via the average of a non-local functional of the initial data. For the viscous Burgers’ equation, however, the formulation is simpler. Explicitly, consider the stochastic flow

\[ dX_t = u_t(X_t) + \sqrt{2\nu} dW_t \]

with initial data \( X_0(a) = a \). Here \( W \) denotes a standard 1D Wiener process. If we require that the velocity \( u \) satisfies

\[ u_t = E \left[ u_0 \circ (X_t^{-1}) \right] \]

where \( E \) denotes the expected value with respect to the Wiener measure, then \( u \) satisfies\(^1\) the viscous Burgers’ equation (1.1) and initial data \( u_0 \). We clarify that in (1.3) and subsequently, \( X_t^{-1} \) denotes the spatial inverse of the stochastic flow \( X_t \). Namely, we know [17, Theorems 4.5.1, 4.6.5] that for regular drifts \( u \), the flow \( X_t \) has a modification which is a stochastic flow of diffeomorphisms of \( \mathbb{R} \). Replacing \( X_t \) with this modification if necessary, \( X_t^{-1} \) denotes the unique process such that \( X_t(X_t^{-1}(x)) = x \) surely for all \( x \in \mathbb{R} \), and \( X_t^{-1}(X_t(a)) = a \) surely for all \( a \).

Observe that when \( \nu = 0 \), the system (1.2)–(1.3) is exactly the method of characteristics for the inviscid Burgers’ equation. Indeed trajectories of the flow \( X_t \) are now characteristics, and equation (1.3) states that the velocity is transported along characteristics. Thus, the \( \nu > 0 \) case could be viewed as a stochastic generalization of the method of characteristics: we transport the initial data along noisy characteristics, and then average with respect to the Wiener measure.

The usual Monte-Carlo method of solving (1.2)–(1.3) numerically [19, 20] is to replace the flow \( X_t \) with \( N \) different copies \( X_{i,N}^t \), each driven by an independent Wiener process \( W_i \), and replace the expected value in (1.3) by the empirical mean:

\[ u_t = \frac{1}{N} \sum_{i=1}^{N} u_0 \circ A_{i,N}^t, \]

where \( u_0 \) is the given initial data, \( W_i \) a sequence of independent Wiener processes and \( \nu > 0 \) the viscosity. As before, \( (X_{i,N}^t)^{-1} \) denotes the spatial inverse of \( X_{i,N}^t \). Throughout this paper, with the exception of Section 3, we impose periodic boundary conditions on the above, and assume the initial data is periodic.

For the Navier-Stokes equations, the particle system in [11] involves using a higher dimensional Wiener process, and replacing (1.7) with the average of vorticity transport and the Biot-Savart law:

\[ \omega_t^N = E \left[ \left( (\nabla X_{i,N}^t) \omega_0 \right) \circ A_{i,N}^t \right], \]

\[ u_t^N = (-\Delta)^{-1} \nabla \times \omega_t^N. \]

\(^1\)This is only valid for spatially periodic or decay at infinity boundary conditions.
where $\omega_0 = \nabla \times u_0$ is the initial vorticity. In [11], the authors considered the system (1.4)–(1.6) & (1.8)–(1.9), with spatially periodic boundary conditions, and proved global existence in two dimensions, local existence in three dimensions, convergence to the correct limit as $N \to \infty$, and described the asymptotic behaviour for fixed $N$ as $t \to \infty$.

Surprisingly, the techniques of [11] fail for the particle system for the Burgers’ equation (the system (1.4)–(1.7)). Indeed, preliminary numerical simulations indicate that the system (1.4)–(1.7) shocks almost surely, in time independent of $N$. We provide a class of initial data for which we can prove (1.4)–(1.7) shocks almost surely. We, however, we are unable to analytically prove that the shock time is independent of $N$.

One heuristic explanation for the shock is as follows: This particle system (1.4)–(1.7) is dissipative only for short time [11, Theorem 5.2]. Once the system (1.4)–(1.7) stops dissipating energy, the growth from the non-linear term should force the system to inherit properties of the inviscid Burgers’ equation, which shocks if the initial data is not monotonically non-decreasing.

We remark that the particle system for the Navier-Stokes equations (the system (1.4)–(1.6) & (1.8)–(1.9)) also dissipates energy only for short time [11, Figure 1 and Theorem 5.2]. In contrast however, no dissipation is required to prove 2D global existence for this system [11, Theorem 3.5]. This is because (1.8)–(1.9) are structurally similar to Euler equations, for which 2D global existence is well known [22] (see also [2, 18]). In contrast, the particle system (1.4)–(1.7) is structurally similar to the inviscid Burgers’ equation which is known to shock in finite time.

The natural approach one would expect to use ‘overcoming’ the shocks in (1.4)–(1.7), would be to continue the system past shocks as weak solutions using an analogue of the Rankine-Hugoniot condition [7, Section 3.4.1], and then prove that as $N \to \infty$, these weak solutions converge to the smooth solutions of the Burgers’ equation. This approach, however, is impossible to use as the stochastic PDE satisfied by $u^N$ involves second order terms, for which the classical techniques [7, Section 3.4.1] will not work.

While the system (1.4)–(1.7) can not be continued past shocks, the shocks can (surprisingly!) be “avoided with large probability” by resetting the Lagrangian maps. This is the main content of this paper. Namely, suppose we solve (1.4)–(1.7) for short time $\delta_t$, and then replace the initial data with $u^N_{k\delta_t}$, and restart the system (1.4)–(1.7) with this new initial data. Our main theorem shows that, if we repeat this procedure often enough, then we can avoid shocks on an arbitrarily large time interval, with probability arbitrarily close to 1.

Explicitly, consider the system

(1.10) \[ dX^{i,N}_{k\delta_t,t} = u^N_i (X^{i,N}_{k\delta_t,t}) dt + \sqrt{2\nu} dW^i_t, \]

(1.11) \[ X^{i,N}_{k\delta_t,k\delta_t}(a) = a, \]

(1.12) \[ A^{i,N}_{k\delta_t,t} = (X^{i,N}_{k\delta_t,t})^{-1}, \]

(1.13) \[ u^N_i = \frac{1}{N} \sum_{i=1}^{N} u^N_i \circ A^{i,N}_{k\delta_t,t}, \]

where $k \in \mathbb{N}$, and $t$ is always assumed to be in the interval $(k\delta_t, (k+1)\delta_t)$.
If $\delta_t$ is small enough, we show that solutions to this system exist on arbitrarily large time intervals, with probability arbitrarily close to one. Once existence with large probability is established, it is easy to show that as $N \to \infty$ these solutions converge to the smooth solutions of the viscous Burgers’ equation.

Before proceeding further, we remark that the fact that the shocks can be avoided by resetting is doubly unexpected! Firstly, we know that the inviscid Burgers’ equations need to be regularized in order for them to have smooth solutions. The resetting procedure above should morally provide no regularization, as explained in Section 2. Secondly, the system (1.2)–(1.3) is Markovian; if we reset it at regular intervals (as above), then the new solution obtained will be no different from the original solution without resetting.

Fortunately, the system (1.4)–(1.7) is not Markovian, and if we reset often enough, the generic short time dissipative effect is strong enough to overcome the nonlinear growth, and with large probability prevents the formation of shocks. We observed numerically that even large resetting time $\delta_t$ (i.e. comparable to half the shock time of the inviscid Burgers’ system) is enough to ensure that the system (1.10)–(1.13) is globally well posed. With the techniques in this paper however, we are only able to prove a global existence result for (1.10)–(1.13) when $\delta_t$ is small. The question for large $\delta_t$ remains open, and cannot be addressed using techniques in this paper.

Finally, we mention that our technique can be used to show global existence of the analogue of (1.10)–(1.13) for the Navier-Stokes equations in two dimensions. As this is already known [11], without resetting, we do not carry out the details here.

One interesting application would be to the 3-dimensional Navier-Stokes equations. There are numerous results showing global existence of solutions to the Navier-Stokes equations with small initial data. One new, interesting question that can be asked in this framework is the existence of solutions for arbitrary initial data, which are time global for some small (non-zero) probability. The McKean-Vlasov type nonlinearity prevents us from asking this question for the stochastic Lagrangian formulation for the Navier-Stokes equations [4, equations (2.3)–(2.6)]. The repeatedly reset version of (1.4)–(1.6) & (1.8)–(1.9) is free of the McKean-Vlasov nonlinearity, and is dissipative, making it a better candidate for small probability time global solutions. Unfortunately, there are obstructions in proving this result directly with the techniques used here, and we are working on addressing this issue.

The plan of this paper is as follows: In Section 2 we establish our notational convention, and prove our main theorem. This proof relies on a few lemmas, the proofs of which we postpone to Sections 4, 5, and 6. In Section 3 we provide an example showing that without resetting, the system (1.4)–(1.7) shocks almost surely. As mentioned earlier, once global existence is established the question of convergence as $N \to \infty$ is easily handled. We conclude the paper by studying this in Section 7.
2. The main theorem and its proof.

Throughout this paper, we assume $(\Omega, \Sigma, P)$ is a probability space and use $E$ to denote the expected value with respect to the probability measure $P$. Let $N \geq 2$ be a natural number (which will be fixed throughout this paper), $\{F_t\}_{t \geq 0}$ be a filtration satisfying the usual conditions on $\Omega$, and $W_t^1, \ldots, W_t^N$ be $N$ independent Wiener processes adapted to the filtration $F_t$. We assume subsequently, without loss of generality, that $\nu = \frac{1}{2}$.

We use $C^k(\mathbb{T})$, to denote the space of all periodic functions on $\mathbb{R}$ (with period 1) which have $k$ continuous derivatives. We use $L^p(\mathbb{T})$, $H^s(\mathbb{T})$ to be the Lebesgue $p$-space, and the Sobolev space of order $s$ respectively, consisting of periodic functions. When writing norms of functions in these spaces, we drop $\mathbb{T}$ for generality, that is right continuous, and the Sobolev space of order $s$ respectively, consisting of periodic functions. When writing norms of functions in these spaces, we drop $\mathbb{T}$. For instance we use the notation $\|u\|_{H^s}$ to denote the $H^s(\mathbb{T})$ norm of $u$.

We use a calligraphic script to denote the analogous spaces for processes, on random time intervals. Namely, given $t_0 \geq 0$, and a stopping time $\tau$ such that $\tau \geq t_0$ almost surely, we define

$$C^k([t_0, \tau]; \mathbb{T}) = \{ u \mid u \in C^0([t_0, \tau]; C^k(\mathbb{T})) \text{ a.s.}, \text{ and } u_{\tau \wedge t} \text{ is } F_t \text{ adapted} \},$$

$$\mathcal{L}^p([t_0, \tau]; \mathbb{T}) = \{ u \mid u \in C^0([t_0, \tau]; L^p(\mathbb{T})) \text{ a.s.}, \text{ and } u_{\tau \wedge t} \text{ is } F_t \text{ adapted} \},$$

$$\mathcal{H}^s([t_0, \tau]; \mathbb{T}) = \{ u \mid u \in C^0([t_0, \tau]; H^s(\mathbb{T})) \text{ a.s.}, \text{ and } u_{\tau \wedge t} \text{ is } F_t \text{ adapted} \},$$

where we use the abbreviation ‘a.s.’ for almost surely. For convenience, if $\tau$ is any stopping time, not necessarily greater than or equal to $t_0$, we define $C^k([t_0, \tau]; \mathbb{T}) = C^k([t_0, \tau \vee t_0]; \mathbb{T})$, and similarly for $\mathcal{H}^s$, $\mathcal{L}^p$. To avoid confusion with the $C^k(\mathbb{T})$ norms, we explicitly use

$$\sup_{\omega \in \Omega} \sup_{t_0 \leq t \leq \tau} \|u_t(\omega)\|_{C^k}$$

to denote the $C^k([t_0, \tau]; \mathbb{T})$ norm of $u$.

We clarify, $u \in C^k([t_0, \tau]; \mathbb{T})$ means that there exists an event $\Omega' \subset \Omega$ with $P(\Omega') = 1$ such that $\forall \omega \in \Omega'$, $t \in [t_0, \tau(\omega)]$, $u_t(\omega) \in C^k(\mathbb{T})$ and $u_t(\omega)$ is continuous in $t$. Further, $\forall t \geq t_0$, $u_{\tau \wedge t}$ is $F_t$-measurable. In words, $C^k([t_0, \tau]; \mathbb{T})$ is the set of all processes which have a $C^k(\mathbb{T})$ valued, continuous paths modification and are defined on the random interval $[t_0, \tau]$.

Note that our spaces involve processes which are continuous in time almost surely, and we are not interested in quantifying any further regularity with respect to time. When the regularity in time needs to be quantified, the definition the analogous spaces is not as elementary (see for instance [14]).

Our main theorem shows given any arbitrarily large $T$, we can make our resetting time $\delta_t$ small enough so that a regular solution to (1.10)–(1.13) exists up to time $T$ with probability arbitrarily close to 1. In order to formulate our Theorem precisely, we will need to define the notion of solutions to the reset system (1.10)–(1.13) with respect to a stopping time. This is our next definition.

**Definition 2.1.** Let $t_0 \geq 0$, $\tau$ be a spatially independent stopping time such that $\tau \geq t_0$ almost surely, and $u_{t_0}$ be a $C^1(\mathbb{T})$ valued $F_{t_0}$ measurable random random variable. Suppose $u \in C^1([t_0, \tau]; \mathbb{T})$ is a unique fixed point of the system

$$(2.1) \quad X^{i,N}_{t_0,t}(a) = a + \int_{t_0}^{\tau \wedge t} u_s(X^{i,N}_{t_0,s}) \, ds + \int_{t_0}^{\tau \wedge t} dW^i_s$$

---

2By ‘usual conditions’ we mean that the filtration $F_t$ is right continuous, and $F_0$ is $P$-complete.
Theorem 2.3.

Let $N > 1$, $T > 0$, $\varepsilon > 0$, $s > 6 + \frac{1}{2}$, and suppose $u_0 \in H^s(T)$. Then there exists $\delta_T = \delta_T(T, \varepsilon, s, \|u_0\|_{H^s})$, independent of $N$, such that for all $\delta_i < \delta_T$, there exists a spatially independent stopping time $\tau$ such that $P(\tau > T) > 1 - \varepsilon$ and the process $u_t$ defined by \((2.4)\) (with $t_0 = 0$) is in the space $C^s([0, \tau]; T)$.

\footnote{The theorem, and proof, remain unchanged if we instead assume that $u_0$ is a $H^s(T)$ valued, $\mathcal{F}_0$-measurable bounded random variable.}

\begin{equation}
A_{t_0, t}^{i, N} = \left(X_{t_0, t}^{i, N}\right)^{-1}
\end{equation}

\begin{equation}
u_t = \frac{1}{N} \sum_{i=1}^{N} u_{t_0} \circ A_{t_0, t}^{i, N}.
\end{equation}

Then we define

\begin{equation}
S_{t_0, t}^{N, \tau} u_{t_0} = u_t.
\end{equation}

For convenience, we adopt the convention that if $\tau$ is any stopping time (not necessarily satisfying $\tau \geq t_0$), we define $S_{t_0, t}^{N, \tau} u_{t_0} = S_{t_0, t}^{N, t_0 \vee t_0} u_{t_0}$.

Remark. Note that it is essential to assume $\tau$ does not depend on the spatial variable, as in this case if the drift $u$ is spatially regular, then the process $X_{t_0, t}^{i, N}$ admits a modification which is a stochastic flow of diffeomorphisms. Hence the spatial inverse $A_{t_0, t}^{i, N} = (X_{t_0, t}^{i, N})^{-1}$ is well defined. We will subsequently always assume our stopping times are spatially independent.

Remark. In Lemma \(2.7\), we will show that $S_{t_0, t}^{N, \tau}$ is well defined. Namely, if $k \geq 1$, and $u_{t_0} \in C^k(T)$ is $\mathcal{F}_{t_0}$-measurable, then Lemma \(2.7\) shows that there exists a (deterministic) time $t_1 > t_0$ such that the process $u$ defined by $u_t = S_{t_0, t}^{N, t_0} u_{t_0}$ belongs to $C^k([t_0, t_1]; T)$.

Remark 2.2. Note that we can view the dependence of the operator $S_{t_0, t}^{N, \tau}$ on the stopping time $\tau$ as a dependence only through the time interval of definition. Indeed, for any fixed $\delta_i$ and stopping time $\tau$, $C^1([0, \tau]; T)$ solutions of \((2.1) - (2.3)\) are unique up to indistinguishability. This follows immediately from a standard Gronwall type argument, and we therefore omit the proof. Strong uniqueness implies that the operator $S_{t_0, t}^{N, \tau}$ satisfies a compatibility condition: for $t_0 \geq 0$, consider two stopping times $\tau_1, \tau_2 \geq t_0$ such that $u_{t}^{1} := S_{t_0, t}^{N, \tau_1} u_{t_0} \in C^1([0, \tau_1]; T)$ and $u_{t}^{2} := S_{t_0, t}^{N, \tau_2} u_{t_0} \in C^1([0, \tau_2]; T)$, then $u_{t}^{2} = u_{t}^{1}$ before $\tau_1 \wedge \tau_2$. That is $u_{t}^{2}$ has a modification such that for all $t \geq t_0$, $u_{t \wedge \tau_1 \wedge \tau_2}^{2} = u_{t \wedge \tau_1 \wedge \tau_2}^{1}$. Thus, when the time interval of definition is clear, we sometimes omit the stopping time $\tau$ as a superscript of our operator $S$.

For notational convenience, we omit the first superscript $N$ for the remainder of this section. Given a (spatially independent) stopping time $\tau$, a deterministic starting time $t_0 \geq 0$, a $C^1(T)$ valued $\mathcal{F}_{t_0}$-measurable initial data $u_{t_0}$, and a resetting time $\delta_i > 0$ small enough, we can define a $C^1([t_0, \tau]; T)$ solution of the (stopped) system \((1.10) - (1.13)\) iteratively by

\begin{equation}
\begin{cases}
\begin{align*}
\dot{u}_t &= u_{t_0} \quad &\text{when } t = t_0, \\
\dot{u}_t^k &= S_{t_0 + k\delta_i, t}^\tau u_{t_0 + k\delta_i} &\text{whenever } t \in (t_0 + k\delta_i, t_0 + (k + 1)\delta_i) \text{ for some } k \in \mathbb{N} \cup \{0\}.
\end{align*}
\end{cases}
\end{equation}
The regularizing effects of resetting

Remark 2.4. The compatibility condition in Remark 2.2 allows us to discuss the notion of a maximal stopping time $\tau_{\text{max}}$ for which the iterative procedure in (2.4) is well defined. Consequently, Theorem 2.3 will show that this maximal stopping time $\tau_{\text{max}}$ is in fact at least $T$ with probability at least $1 - \varepsilon$.

We emphasize that the operator $S_{t_0,t}$ is not a smoothing operator, which as mentioned earlier is part of the reason why Theorem 2.3 is surprising. We can see $S_{t_0,t}$ is not smoothing from the fact that (4.3), the stochastic partial differential equation (SPDE) satisfied by $u_t = S_{0,t}u_0$ is not dissipative \cite{14}. One can immediately verify this as the diffusive term in (4.3) does not necessarily dominate the noise.

Another (perhaps more intuitive) way of understanding the regularity properties of $S_{t_0,t}$ is via time splitting. The $S_{t_0,t}$ can be time split into two parts: $\bar{S}_{1,t_0}$, the nonlinear solution operator associated with the inviscid Burgers’ equations, and $\bar{S}_{2,t_0}$ the operator corresponding to resetting. By considering time split version of (1.10), one can see that $\bar{S}_{2,t_0}$ exactly corresponds to the operator

\begin{equation}
\bar{S}_{2,t_0} f(x) = \frac{1}{N} \sum_{j=1}^{N} f \left( x - \left( W_{t_0+\delta t}^j - W_{t_0}^j \right) \right).
\end{equation}

The operator $\bar{S}_{1,t_0}$ causes growth on the Fourier modes. It is well known that the damping provided by $\nu \partial_x^2$ for any $\nu > 0$, is enough to overcome this growth, and this gives us global existence for the viscous Burgers’ equations for any strictly positive viscosity. Thus if the operator $\bar{S}_{2,t_0}$ provides damping comparable to $\nu \partial_x^2$, then the usual methods can be used to prove Theorem 2.3. However, the operator norm of $\bar{S}_{2,t_0}$ is exactly 1 (surely) in all Sobolev and Hölder spaces. This is the main difficulty in proving Theorem 2.3.

We overcome this difficulty by considering the limit $v := \lim_{\delta t \to 0} u^{\delta t}$. It turns out that $v$ satisfies a dissipative SPDE, and if the initial data is regular enough we obtain convergence in a strong norm of $u^{\delta t}$ to $v$. This is the key to the proof of Theorem 2.3 and is formulated below.

Lemma 2.5 (Key Lemma). Let $N > 1$, $\beta \in \mathbb{N} \cup \{0\}$, $T_0 > t_0 \geq 0$, and $\tau$ be a (spatially independent) stopping time. Let $u_{t_0}$ be a $C^{4+\beta}(\mathbb{T})$ valued $\mathcal{F}_{t_0}$-measurable random variable, and $u^{\delta} \in C^{4+\beta}([t_0, \tau]; \mathbb{T})$ be defined by (2.4). Let $v \in C^{4+\beta}([t_0, \tau]; \mathbb{T})$ be the solution of the SPDE

\begin{equation}
dv_t + v_t \partial_x v_t \, dt - \frac{1}{2} \partial_x^2 v_t \, dt + \frac{\partial_x v_t}{N} \sum_{j=1}^{N} dW_t^j = 0.
\end{equation}

\textit{The proof of this lemma never uses the assumption $N > 1$, and is valid even for $N = 1$. However, for $N = 1$, Lemma 2.5 is vacuously true as assumptions (2.7) and (2.8) will never be satisfied for non-constant initial data.}
with initial data $v|_{t=t_0} = u_{t_0}$, and spatially periodic boundary conditions. Let $\tau_0 = (\tau \lor t_0) \land T_0$, and $U$ be a constant such that

\[
\sup_{t_0 \leq t \leq \tau_0} \| u_t^\delta \|_{C^{4+\beta}} \leq U \quad \text{a.s.,}
\]

and let $u_t^\delta = u_t^\delta_{\tau_\Lambda} - v_{\tau_\Lambda}$. Then there exists a constant $C = C(\beta, U, T_0)$, independent of $N$, $\delta_t$, and $\tau$, such that

\[
\sup_{t_0 \leq t \leq T_0} \mathbb{E} \left\| u_t^\delta \right\|_{H^\beta}^2 \leq C\delta_t^{1/2}.
\]

Our main interest in this lemma will be for $\beta = 2$, as it will enable us to obtain a $C^1([t_0, T_0]; T)$ bound on $u$ from a $C^1([t_0, T_0]; T)$ bound on $v$. A $C^1(T)$ bound is all that is needed to continue a solution locally, thus controlling the $C^1(T)$ norm of $u$ with large probability, independent of $\delta_t$, will prove our theorem. Since (2.6) is dissipative, uniform in time bounds of strong norms of $v$ are readily obtained.

**Lemma 2.6.** Let $N > 1$, $s \in \mathbb{N}$, $u_0 \in H^s(\mathbb{T})$. There exists a process $v \in \mathcal{H}^s([0, \infty); \mathbb{T})$ which is a solution to the SPDE (2.6) with initial data $u_0$ and periodic boundary conditions. Further, there exists a constant $V_s = V_s(s, \|u_0\|_{H^s})$ such that

\[
\sup_{t \geq 0} \| v_t \|_{H^s} \leq V_s
\]

almost surely.

We remark that (2.10) is an almost sure bound on a strong norm of $v$. The reason we are able to obtain almost sure bounds is because if we “multiply by $v$ and integrate by parts” (or more precisely, apply Itô’s formula to $\| v_t \|_{L^2}$), we obtain an equation with no martingale part! This is carried out in detail in Section 5.

Lemmas 2.5 and 2.6 will now allow uniform in time control of a strong norm of $u^\delta$. The only remaining ingredient is to obtain a $C^1(T)$ local existence result, and to ensure that the inequality (2.7) is satisfied uniformly in $\delta_t$.

**Lemma 2.7.** Suppose $u_{t_0}$ is a $C^1(T)$ valued $F_{t_0}$-measurable random variable such that there exists a constant $U_0 > 0$ such that $\| u_{t_0} \|_{C^1} \leq U_0$ almost surely. There exists $T_0 = T_0(U_0) > t_0$ and a process $u^\delta_t \in C^1([t_0, T_0]; T)$ such that $u^\delta_t$ is a solution to (2.4) with $\tau = T_0$.

If further for some $n \in \mathbb{N}$, $u_{t_0}$ is a $C^n(T)$ valued $F_{t_0}$-measurable random variable, and there exists a constant $U_n^0 > 0$ such that $\| u_{t_0} \|_{C^n} \leq U_n^0$ almost surely, then $u^\delta_t$ is $C^n([0, T]; T)$, and further there exists a constant $U_n = U_n(U_n^0, n)$, independent of $N$ and $\delta_t$, such that

\[
\sup_{t_0 \leq t \leq T_0} \| u_t^\delta \|_{C^n} \leq U_n \quad \text{a.s.}
\]

for all $\delta_t < T_0$.

\footnote{The assumptions (2.7), (2.8) can be weakened slightly at the expense of a lengthier, more technical proof. The weakened assumptions however, still require more than $\beta$ derivatives. While replacing (2.7) (2.8) with a condition involving only $\beta$ derivatives would be of sufficient interest to warrant a more technical proof, reducing $4 + \beta$ to $4 + \beta - \varepsilon$ only obscures the heart of the matter. Since sufficient regularity on our initial data will guarantee (2.7), (2.8) anyway, we assume they hold and avoid unnecessary technicalities.}
Remark. The existence time $T_0$ above only depends on a the $C^1(T)$ norm of the initial data. However, on the existence interval, any additional regularity of the initial data is preserved.

We are now ready to prove the main theorem. (The Lemmas 2.5, 2.6 and 2.7 will be proved in Sections 4, 5 and 6 respectively.)

**Proof of Theorem 2.3.** Let $\delta_T > 0$ be a small time, to be specified later, and let $\delta_t \in (0, \delta_T)$ be arbitrary. Given a stopping time $\tau$, we define the operator $\mathcal{F}_{\delta_t, \tau}^{\delta_t, \tau}$ by

$$
\mathcal{F}_{\delta_t, \tau}^{\delta_t, \tau} = S_{\delta_t, \tau}^r \circ S_{(k-1)\delta_t, k\delta_t}^r \circ \cdots \circ S_{(m+1)\delta_t, (m+2)\delta_t}^r \circ S_{m\delta_t, (m+1)\delta_t}^r,
$$

where $k \in \mathbb{N}$ is such that $k\delta_t < t \leq (k+1)\delta_t$.

Let $v_t$ be the solution of (2.6). By Lemma 2.6 and the Sobolev embedding theorem there is a constant $V_1$, such that

$$
\sup_{t \geq 0} \|v_t\|_{C^1} \leq V_1
$$

almost surely. Let $T_0 = T_0(2V_1)$ be the local existence time in Lemma 2.7, namely, for any initial data $u_0$ with $\|u_0\|_{C^1} \leq 2V_1$, and for any $\delta_t < T_0$, the process $\mathcal{F}_{\delta_t, T_0}^r u_0$ is $C^1([0, T_0]; T)$. Without loss of generality we can assume that $T_0$ is an integer multiple of $\delta_t$.

Note that our assumption $u_0 \in H^{13/2+}$, Lemma 2.6 and the Sobolev embedding theorem imply the assumption (2.8) is valid for $\beta = 2$ (in this case, the supremum can in fact be taken over all $t \in \mathbb{R}$). Similarly Lemma 2.7 guarantees that the assumption (2.7) is valid for $\beta = 2$ and all $\delta_t < T_0$. Thus Lemma 2.5 can be applied.

Let $\Omega_1$ be the event \{\$u_{T_0}^{\delta_t} \|_{C^1} \leq 2V_1\$. Then

$$
P(\Omega_1) \geq P \left( \left\| u_{T_0}^{\delta_t} - v_{T_0} \right\|_{C^1} \leq V_1 \right)
$$

$$
\geq P \left( \left\| u_{T_0}^{\delta_t} - v_{T_0} \right\|_{H^2} \leq \frac{V_1}{c_1} \right) \quad \text{[Sobolev embedding]}
$$

$$
\geq 1 - \frac{c_1^2}{V_1^2} E \left( \left\| u_{T_0}^{\delta_t} - v_{T_0} \right\|_{H^2}^2 \right) \quad \text{[Chebyshev’s inequality]}
$$

$$
\geq 1 - \frac{C\delta_t^{1/2}}{V_1^2} \quad \text{[Lemma 2.5]},
$$

$$
\geq 1 - \frac{C\delta_t^{1/2}}{V_1^2},
$$

where the constant $c_1$ above is the constant arising in the Sobolev embedding theorem. An appropriate choice of $\delta_T$ will make $P(\Omega_1)$ arbitrarily close to 1. We clarify that while our bound on $P(\Omega_1)$ depends only on $\delta_T$, the event $\Omega_1$ depends on $\delta_t$.

We define a stopping time $\tau_1$ by

$$
\tau_1(\omega) = \begin{cases} 
T_0 & \text{if } \omega \notin \Omega_1, \\
2T_0 & \text{if } \omega \in \Omega_1.
\end{cases}
$$

Note that by Remark 2.2 we have $\mathcal{F}_{\delta_t, \tau_1}^r u_0 = \mathcal{F}_{\delta_t, T_0}^r u_0$ for all $t \in [0, T_0]$. Thus, when we write $S$ we really mean the solution operator of (2.4).
Proposition 2.8. \(pN\)ToInfinity
event of almost full probability. By Lemma 2.7 and Remark 2.2. Now, for \(\omega\)work with (1.4)–(1.7) on \(R\)functions on \(R\)the solution of the viscous Burgers’ equation (1.1) \(S\)defined and in \(as long as either side is defined. We claim that the right hand side above is well-defined and in \(C^6([0, \tau_1]; T)\).

We see this as follows: First for \(t \in [0, T_0]\), this is true by Lemma 2.7 and Remark 2.2. Now, for \(\omega \notin \Omega_1\) and any \(t \in [T_0, 2T_0]\), \(S_{T_0, t}^{\delta, \tau_1}\) is just the identity operator. Further for almost every \(\omega \in \Omega_1\) we have \(S_{0, T_0}^{\delta, T_0} u_0(\omega) = u_{T_0}^{\delta} (\omega) \in C^6(T)\) and \(\|u_{T_0}^{\delta}(\omega)\|_{C^1} \leq 2V_1\). Thus for almost any \(\omega \in \Omega_1\), and for every \(t \in [T_0, 2T_0]\), \(S_{T_0, t}^{\delta, \tau_1} u_{T_0}^{\delta}(\omega) \in C^6(T)\) by Lemma 2.7.

Using Sobolev embedding, Chebyshev’s inequality and Lemma 2.5 as above, we can find an event \(\Omega_2 \subset \Omega_1\) such that \(P(\Omega_2)\) is arbitrarily close to \(P(\Omega_1)\). As before we define a stopping time \(\tau_2\) by

\[
\tau_2(\omega) = \begin{cases} \tau_1(\omega) & \text{if } \omega \notin \Omega_2, \\ 3T_0 & \text{if } \omega \in \Omega_2. \end{cases}
\]

and the solution \(u_{\tau_2}^{\delta} = S_{0, \tau_2}^{\delta, \tau_2} u_0 \in C^6([0, \tau_3]; T)\). A finite iteration will complete the proof. \(\square\)

Finally we address the question of \(N \to \infty\). For this purpose, we re-introduce the superscript of \(N\) to indicate the dependence on \(N\) of the process considered. Using techniques similar to [11], we show that the solution \(u^N\) of (2.6) converges to the solution of the viscous Burgers’ equation as \(N \to \infty\).

**Proposition 2.8.** Let \(u^N\) be the solution of (2.6) with initial data \(u_0\), and \(u_1^N\) be the solution of the viscous Burgers’ equation (1.1) with the same initial data. If \(u_0 \in H^s, s > \frac{3}{2}\), then for any \(T > 0\), there exists a constant \(C = C(T, s, \|u_0\|_{H^s})\) such that

\[
\sup_{t \in [0, T]} E \left\| u_1^N - u^N \right\|_{L^2}^2 \leq C. 
\]

We prove proposition 2.8 in Section 7. We conclude by remarking that Proposition 2.8, Lemma 2.3 and an argument similar to the proof of Theorem 2.3 will show that for small enough \(\delta\), as \(N \to \infty\), \(u^N_{N, \delta}\) converges to the same limit on an event of almost full probability.

**3. Almost sure existence of shocks without resetting**

In this section we show that the system (1.4)–(1.7) develops shocks almost surely, for any \(N\). The existence of shocks is simpler to prove if we work with monotone functions on \(\mathbb{R}\), instead of periodic functions, and thus for this section only, we will work with (1.4)–(1.7) on \(\mathbb{R}\) instead of on \(T\).

Let \(\tau\) be a (spatially-independent) stopping time, and we interpret \(C^4([0, \tau]; \mathbb{R})\) solutions to (1.4)–(1.7), in the natural way (analogous to (2.1)–(2.3)). The main result of this section shows that even if we stop ‘bad’ realizations of \(N\), we can never continue solutions past the time \(\delta u_0 \|_{L^\infty}\), unless we introduce a regularizing mechanism.
Proposition 3.1. Suppose \( u_0 \in C^1(\mathbb{R}) \) is a decreasing function, and let \( u \) be a \( C^1([0, \tau']; \mathbb{R}) \) a solution of (1.4)–(1.7) with initial data \( u_0 \). Then, almost surely,

\[
\tau' < \frac{N}{\|\partial_x u_0\|_{L^\infty}}.
\]

Remark 3.2. The numerically observed shock time, in the periodic case, is independent of \( N \), and it is of the order \( 1/N \) with large probability. This indicates our bound (3.1) is far from optimal.

Remark 3.3. One can show\(^6\) that as \( N \to \infty \) the solution to (1.4)–(1.7) approaches the solution to (1.1) at a rate of \( 1/\sqrt{N} \). However, it is well known that the solution to (1.1) is smooth for all time and no shock develops, provided the initial data is for instance \( C^1 \) and bounded \([7]\).

The numerics mentioned in Remark 3.2, however, indicate that no matter how large \( N \) is, the system (1.4)–(1.7) will only be a good approximation to the true solution of (1.1) for short time, in the order of \( 1/N \).

Remark 3.4. Monotonicity of the initial data \( u_0 \) is precisely the condition that constrained us to work on the line instead of on the torus. Specifically, the assumption \( \partial_x u_0(x) < 0 \) for arbitrary \( x \in \mathbb{R} \) simplifies the proof of 3.1 considerably. Numerics, however, indicate that this monotonicity assumption is redundant, and (1.4)–(1.7) develops shocks for arbitrary (periodic) initial data.

Proof of Proposition 3.1. Assume for simplicity, and without loss of generality, that \( \|\partial_x u_0\|_{L^\infty} = -\partial_x u_0(0) = 1 \). Let the stopping time \( \tau \) be the first time \( t \leq \tau' \)

\[ \tau = \tau' \wedge \inf \{ t \mid \partial_x X_{t,N}^1(0) = 0 \}. \]

We will first show that, \( \tau \leq N \), almost surely. Differentiating (1.4) in space gives

\[
d(\partial_x X_{t,N}^1) = \partial_x u_t^N \bigg|_{X_{t,N}^1} \partial_x X_{t,N}^1 \, dt,
\]

for \( t < \tau' \), almost surely. Here, our notation \( \partial_x u_t^N \big|_{X_{t,N}^1} \) means

\[ \partial_x u_t^N \big|_{X_{t,N}^1}(x) = \partial_x u_t^N (X_{t,N}^1(x)). \]

Differentiating equation (1.7) in space, we obtain

\[
\partial_x u_t^N \big|_{X_{t,N}^1} = \frac{1}{N} \sum_{i=1}^{N} \partial_x (u_0 \circ A_{t,N}^i) \bigg|_{X_{t,N}^1}
\]

for \( t < \tau' \) almost surely. Since by the chain rule,

\[
\partial_x A_{t,N}^1 \big|_{X_{t,N}^1} \partial_x X_{t,N}^1 = \partial_x \left( A_{t,N}^1 \circ X_{t,N}^1 \right) = 1,
\]

\[ \text{See for instance [11, Theorem 4.1], where the analogous result is proved for the Navier-Stokes equations.} \]
multiplying equation (3.3) by $\partial_x X_t^{1,N}$ gives

$$
(3.5) \quad \partial_x u_t^N \bigg|_{X_t^{1,N}} = \frac{1}{N} \left[ \partial_x u_0 + \sum_{i=2}^N \partial_x u_0 \bigg|_{A_t^{i,N} \circ X_t^{1,N}} \cdot \partial_x A_t^{i,N} \bigg|_{X_t^{1,N}} \right]
$$

for $t < t'$ almost surely.

Note that for a $C^1$ solution of the system (1.4)–(1.7), for all i, the flow $X_t^{i,N} : \mathbb{R} \to \mathbb{R}$ is homotopic to the identity map via $C^1$ diffeomorphisms of $\mathbb{R}$. The same is true for the inverse inverse $A_t^{i,N}$, and thus $\partial_x A_t^{i,N} \big|_{X_t^{i,N}} > 0$. Finally, since $u_0$ is assumed to be decreasing, we know that $\partial_x u_0 < 0$, and thus equations (3.2) and (3.5) yield

$$
\partial_\tau \partial_x X_t^{1,N}(0) < -\frac{1}{N}
$$

for $t < t'$ almost surely. This (ordinary) differential inequality, along with the fact that $\partial_x X_t^{1,N} = 1$, necessitates $\tau < N$ almost surely.

Now, by definition of $\tau$, and continuity (in time) of $\partial_x X_t^{1,N}$,

$$
(3.6) \quad \lim_{t \to \tau^-} \partial_x X_t^{1,N}(0) = 0
$$

on the event $\{\tau < \tau'\}$. From (3.5) and the chain rule we have

$$
\partial_x u_t^N \bigg|_{X_t^{1,N}} = \frac{1}{N} \left[ \partial_x u_0 + \sum_{i=2}^N \partial_x u_0 \bigg|_{A_t^{i,N} \circ X_t^{1,N}} \cdot \partial_x A_t^{i,N} \bigg|_{X_t^{1,N}} \right]
$$

for $t < \tau'$ almost surely. Note that all the terms on the right have the same sign. Thus if one of these terms approaches $-\infty$, then necessarily the entire right hand side approaches $-\infty$. Equation (3.6) immediately implies the first term approaches $-\infty$ at $x = 0$ on the event $\{\tau < \tau'\}$. Hence, on this event we have

$$
\lim_{t \to \tau^-} \|\partial_x u_t\|_{L^\infty} \geq - \lim_{t \to \tau^-} \partial_x u_t^N(X_t^{1,N}(0)) = \infty,
$$

almost surely. Consequently, if $u \in C^1([0,\tau']; \mathbb{R})$, we must have $P(\tau < \tau') = 0$. Hence $\tau' = \tau < N$ almost surely.

4. Proof of the Key Lemma (Lemma 2.5).

In this section we prove convergence of $u^{\delta_t}$ to $v$ as $\delta_t \to 0$. The basic idea is to show that the velocity in our reset system (1.10)–(1.13) satisfies the limiting SPDE (2.6) with a small error which is controlled as $\delta_t \to 0$.

By shifting time, we may assume without loss of generality that $t_0 = 0$. Further replacing $\tau$ with $\tau \wedge T_0$ if necessary, we may assume $\tau = \tau_0 \leq T_0$. Throughout this section, we adopt the convention that $t_0 = 0$, and $N$, $\beta$, $T_0$, $\tau$, $\tau_0$, $u_0$, $u^{k_i}$, $v$ and $U$ are as in the statement of Lemma 2.5. We also assume the processes $X_t^{1,k_i}$, $A_t^{k_i}$, are all as in (1.10)–(1.13), and for notational convenience, we will omit the $N$ and $\delta_t$ as superscripts throughout this section.

We need a few lemmas before we can prove Lemma 2.5. In our first lemma we determine an SPDE satisfied by $u$ on the interval $(k\delta_t, (k+1)\delta_t]$.

**Lemma 4.1.** We define the process $s^i$ to be the $i^{th}$ summand in (1.13). Explicitly,

$$
u_t^i = \begin{cases} u_0 & \text{for } t = 0, \\ u_{k\delta_t} \circ A_{k\delta_t,t}^{i} & \text{for } t \in (\tau \wedge k\delta_t, \tau \wedge (k+1)\delta_t]. \end{cases}
$$


Proof. From [4,11] (see also [15,21]) we know that when an iteration argument is unnecessary because of assumption (2.7).

Then for all $i \in \{1, \ldots, N\}$, the process $u^i \in C^{4+\beta}([0, \tau]; \mathbb{T})$, and satisfies the SPDE

\begin{align}
(4.2) \quad \chi_{\{\tau \geq k\delta_i\}} \left(u^i_{\tau \wedge t} - u_{k\delta_i}\right) + \int_{\tau \wedge k\delta_i}^{\tau \wedge t} \left(u^i_s \partial_x u^i_s - \frac{1}{2} \partial^2_x u^i_s\right) ds + \
+ \int_{\tau \wedge k\delta_i}^{\tau \wedge t} \partial_x u^i_s dW^i_s = 0.
\end{align}

on the interval $t \in [k\delta_i, (k+1)\delta_i]$. Similarly, the process $u \in C^{4+\beta}([0, \tau]; \mathbb{T})$, and satisfies the SPDE

\begin{align}
(4.3) \quad u_{\tau \wedge t} - u_{\tau \wedge k\delta_i} + \int_{\tau \wedge k\delta_i}^{\tau \wedge t} \left(u_s \partial_x u_s - \frac{1}{2} \partial^2_x u_s\right) ds + \
+ \int_{\tau \wedge k\delta_i}^{\tau \wedge t} \frac{1}{N} \sum_{j=1}^{N} \partial_x u^j_s dW^j_s = 0.
\end{align}

on the interval $t \in [k\delta_i, (k+1)\delta_i]$.

Remark 4.2. A more intuitive, though less precise, way of phrasing the SPDE’s (4.2) and (4.3) would be to say for $t \in (\tau \wedge k\delta_i, \tau \wedge (k+1)\delta_i)$, $u^i$, $u$ satisfy the SPDE’s

\begin{align}
du^i_t + u_t \partial_x u^i_t dt - \frac{1}{2} \partial^2_x u^i_t dt + \partial_x u^i_t dW^i_t = 0, \quad \text{for all } i \in \{1, \ldots, N\}
\end{align}

\begin{align}
du_t + u_t \partial_x u_t dt - \frac{1}{2} \partial^2_x u_t dt + \frac{1}{N} \sum_{j=1}^{N} \partial_x u^j_t dW^j_t = 0
\end{align}

with initial data $u^i_{t=k\delta_i} = u_{k\delta_i}$ and $u_{t=k\delta_i} = u_{k\delta_i}$.

Proof. From [4,11] (see also [15,21]) we know that when $\tau \equiv \infty$, the process $A^i_{k\delta_i}$, satisfies the SPDE

\begin{align}
dA^i_{k\delta_i,t} + u_t \partial_x A^i_{k\delta_i,t} dt - \frac{1}{2} \partial^2_x A^i_{k\delta_i,t} dt + \partial_x A^i_{k\delta_i,t} dW^i_t = 0
\end{align}

on the time interval $(k\delta_i, (k+1)\delta_i]$. Writing down an integral version of this in the presence of a stopping time, equations (4.2) and (4.3) follow immediately from (2.3) and (4.1) by a direct application of Itô’s formula.

To check $u^i, u, u^1, \ldots, u^N \in C^{4+\beta}([0, \tau]; \mathbb{T})$, note that continuity in time is immediate. Further, the spatial regularity of $u$ has already been assumed in the statement of Lemma 2.5. For $u^1, \ldots, u^N$, note that the $\tau$ and the noise are spatially independent in equation (1.10), and it immediately shows that each $X^i_{k\delta_i}$, (and hence each $A^i_{k\delta_i}$, ) is as spatially regular as $u$, which in turn shows that each $u^i \in C([0, \tau]; \mathbb{T})$.

Now we show that with a small error $u$ satisfies the SPDE (2.6) stopped at $\tau$, and obtain bounds on this error. Let $E_{\mathcal{F}_{k\delta_i}} Y$ denote the conditional expectation of $Y$ given $\mathcal{F}_{k\delta_i}$. Given any process $f$, and a stopping time $\tau$, we define the stopped increment $\Delta^\tau_t f$ by

\begin{align}
\Delta^\tau_t f = f_{\tau \wedge (k+1)\delta_i} - f_{\tau \wedge k\delta_i}.
\end{align}

The spatial regularity of $u, u^1, \ldots, u^N$ follows directly from an assumption only on the initial data, and a standard iteration argument. This is contained in Section 6. However for Lemma 4.4, an iteration argument is unnecessary because of assumption (2.7).
Lemma 4.3. Suppose (2.7) holds for some $\beta \in \mathbb{N} \cup \{0\}$, and let $\varepsilon'_k$ be defined by 

\[ \varepsilon'_k = \Delta^*_t u + L_{k\delta_t} \Delta^*_k t + \partial_x u_{k\delta_t} \left( \frac{1}{N} \sum_{j=1}^{N} \Delta [W^3] \right). \]

Then there exists a constant $C = C(\beta, U, T_0)$ (independent of $N, k, \delta_t$ and $\tau$) such that for all $\delta_t \leq T_0$ and $k \leq \frac{T_0}{\delta_t}$, we have

\[ \sup_{x \in \mathbb{T}} E \left| \partial_x \varepsilon'_k(x) \right|^2 \leq C \delta_t^2, \]

\[ \sup_{x \in \mathbb{T}} E \left| \partial_x \varepsilon'_k(x) \right|^2 \leq C \delta_t^3. \]

Remark. Since $u$ and all derivatives of $u$ are a priori uniformly bounded almost surely, the proof of this lemma is straightforward. Without this a priori bound, we would only obtain similar bounds on $E\|\partial_x \varepsilon'_k\|_{L^2}^2$ and $E\|\partial_x \varepsilon'_k\|_{L^2}^2$, which are still sufficient for Lemma 2.5.

Proof of Lemma 4.3. We assume throughout this section that $C$ is a constant only depending on $U$ and $T_0$ which could change from line to line. Note first that assumption (2.7) and equation (2.1) immediately imply that for any $i \in \{1, \ldots, N\}$,

\[ \sup_{0 \leq \tau \leq T_0} \|\partial_x X_i\|_{C^{3+\beta}} \leq C \quad \text{and} \quad \sup_{0 \leq \tau \leq T_0} \|\partial_x A_i\|_{C^{3+\beta}} \leq C \]

almost surely. Now equation (2.3), immediately yields the same bound for $u$, independent of $N$. Thus, making $U$ larger if necessary, we may assume without loss of generality that (2.7) holds for all the processes $u, u^1, u^2, \ldots, u^n$.

For any $k \in \mathbb{N} \cup \{0\}, t \in (k\delta_t, (k + 1)\delta_t], n \leq \beta + 2$, differentiating (4.3) $n$ times gives

\[ \partial^n_x u_{\tau \wedge t} - \partial^n_x u_{\tau \wedge k\delta_t} = -\int_{\tau \wedge k\delta_t}^{\tau \wedge t} \partial^n_x L u_s \, ds - \frac{1}{N} \sum_{j=1}^{N} \int_{\tau \wedge k\delta_t}^{\tau \wedge t} \partial^n_x \partial_x u_{j\delta_t}^n \, dW_s^j. \]

and hence

\[ E \left| \partial^n_x u_{\tau \wedge t} - \partial^n_x u_{\tau \wedge k\delta_t} \right|^2 \leq 2E \left( \int_{\tau \wedge k\delta_t}^{\tau \wedge t} \partial^n_x L u_s \, ds \right)^2 + 2E \left( \frac{1}{N} \sum_{j=1}^{N} \int_{\tau \wedge k\delta_t}^{\tau \wedge t} \partial^n_x \partial_x u_{j\delta_t}^n \, dW_s^j \right)^2. \]

\[ \text{Lemma 4.3.} \]
Thus using (2.7) for both $u$ and $u^i$, for any $t \in (k\delta_t, (k+1)\delta_t]$ we have

$$\sup_{x \in \mathbb{T}} E \left| \partial_x^2 u_{\tau \wedge \tau} (x) - \partial_x^2 u_{\tau \wedge \delta_t} (x) \right|^2 \leq C \left( \delta_t^2 + \frac{1}{N^2} \sum_{j=1}^{N} \delta_t \right) \leq C \delta_t,$$

where as usual the constant $C$ may change from line to line, provided it only depends on $\beta$, $T$ and $T_0$.

Similarly, using (4.2) and the above argument we have

$$\sup_{x \in \mathbb{T}} E \left| \partial_x^2 u_{\tau \wedge \tau} (x) - \partial_x^2 u_{\tau \wedge \delta_t} (x) \right|^2 \leq C \delta_t,$$

for any $t \in (k\delta_t, (k+1)\delta_t]$ and $n \leq 2 + \beta$.

Now, from the definition of $\varepsilon'$ and equation (4.3) we have

$$\varepsilon'_k = - \int_{\tau \wedge \delta_t}^{\tau \wedge (k+1)\delta_t} L u_s \, ds - \frac{1}{N} \sum_{j=1}^{N} \int_{\tau \wedge \delta_t}^{\tau \wedge (k+1)\delta_t} \partial_x u_s^i \, dW_s^j +$$

$$+ L u_{\tau \wedge \delta_t} \Delta^k \delta_t + \partial_x u_{\tau \wedge \delta_t} \left( \frac{1}{N} \sum_{j=1}^{N} \Delta^k W_s^j \right)$$

almost surely. For the Itô integrals in the second term above,

$$E_{\mathcal{F}_{\delta_t}} \left( \int_{\tau \wedge \delta_t}^{\tau \wedge (k+1)\delta_t} \left( \partial_x u_{\tau \wedge \delta_t} - \partial_x u_s^i \right) \, dW_s^j \right) =$$

$$= 0,$$

and hence

$$E \left| E_{\mathcal{F}_{\delta_t}} \partial_x^2 \varepsilon'_k \right|^2 = E \left( \partial_x^2 \int_{\tau \wedge \delta_t}^{\tau \wedge (k+1)\delta_t} \left( L u_s - L u_{\tau \wedge \delta_t} \right) \, ds \right)^2$$

$$\leq \delta_t \int_{\delta_t}^{(k+1)\delta_t} E \left[ \chi_{\{s \leq \tau\}} \chi_{\{s \leq \tau\}} \partial_x^3 (L u_{\tau \wedge \delta_t} - L u_{\tau \wedge \delta_t}) \right]^2 \, ds$$

$$= \delta_t \int_{\delta_t}^{(k+1)\delta_t} E \left[ \chi_{\{s \leq \tau\}} \partial_x^3 (L u_{\tau \wedge \delta_t} - L u_{\tau \wedge \delta_t}) \right]^2 \, ds$$

$$\leq C \delta_t^3,$$

where the last inequality follows from (4.7) with $n = 2 + \beta$. This proves (4.6).

For (4.5), note that the expected value of the square of the first term in (4.9) has already been bounded by $C \delta_t^3 < C \delta_t^2$. For the second term, the Itô isometry gives
Lemma 4.4. Suppose (2.8) holds for some \( \beta \in \mathbb{N} \cup \{0\} \), and let \( \varepsilon_k'' \) be defined by

\[
\varepsilon_k'' = \Delta_k v + L v_{k\delta} + \Delta_k + \partial_x v_{k\delta} \left( \frac{1}{N} \sum_{j=1}^N \Delta_k W^j \right).
\]

Then the bounds (4.5) and (4.6) hold for \( \varepsilon_k'' \).

Proof. First note that

\[
v_{\tau\wedge t} - v_{\tau\wedge k\delta t} = -\int_{\tau\wedge t}^{\tau\wedge k\delta t} L v_s \, ds - \frac{1}{N} \sum_{j=1}^N \int_{\tau\wedge k\delta t}^{\tau\wedge t} \partial_x v_s \, dW_s^j
\]

almost surely. Thus for any \( n \leq 2 + \beta \) and \( t \in [k\delta, (k+1)\delta] \) using (2.8) gives

\[
\sup_{x \in \mathbb{T}} E \left[ |\partial^n_x v_{\tau\wedge t}(x) - \partial^n_x v_{\tau\wedge k\delta t}(x)|^2 \right] \leq C_1 \delta t.
\]

Similar to the derivation of (4.9) we obtain

\[
\varepsilon_k'' = \int_{\tau\wedge (k+1)\delta}^{\tau\wedge (k+1)\delta} (L v_{k\delta} - L v_s) \, ds + \frac{1}{N} \sum_{j=1}^N \int_{\tau\wedge (k+1)\delta}^{\tau\wedge (k+1)\delta} (\partial_x v_{k\delta} - \partial_x v_s) \, dW_s^j
\]

from definition (4.10). The remainder of the proof is now identical to the proof of Lemma 4.3. \( \square \)

We are now ready to prove Lemma 2.5. We remark that, the assumptions (2.7) and (2.8) are stronger than necessary. We only need

\[
\sup_{0 \leq t \leq \tau} \left( |u_t|_{C^{1+\beta}} + |v_t|_{C^{1+\beta}} \right) \leq U, \quad \text{a.s.}
\]

and the bounds on \( \varepsilon', \varepsilon'' \) provided by Lemmas 4.3 and 4.4 above. The proof we provide below depends only on these weaker assumptions.

Proof of Lemma 2.5. Let \( \varepsilon_k = \varepsilon_k' - \varepsilon_k'' \), where \( \varepsilon_k', \varepsilon_k'' \) are defined by Lemmas 4.3 and 4.4 respectively. Using equations (4.5), (4.6) and the corresponding estimates for \( \varepsilon_k'' \), we have

\[
\sup_{x \in \mathbb{T}} E \left| \partial^n_x \varepsilon_k(x) \right|^2 \leq C \delta t^2
\]

and

\[
\sup_{x \in \mathbb{T}} E \left| E_{k\delta} \partial^n_x \varepsilon_k(x) \right|^2 \leq C \delta t^3.
\]
for all $k \leq \frac{T_0}{\beta}$. As before, we assume $C$ is a constant that only depends on $\beta$, $U$ and $T_0$, which may change from line to line. Now, estimates \ref{eqn:epsilon_Sob} and \ref{eqn:epsilon_Given_Fk} imply

\begin{align}
\tag{4.16} E \| \epsilon_k \|^2_{H^\beta} &\leq C \delta_t^2 \\
\tag{4.17} E \left\| E_{F_k}\epsilon_k \right\|^2_{H^\beta} &\leq C \delta_t^3.
\end{align}

For the remainder of the proof we will use the weaker estimates \ref{eqn:epsilon_Sob} and \ref{eqn:epsilon_Given_Fk}. Now, recall $w_t = u_{T_\wedge t} - u_{T_\wedge t}$, and we know $w_0 = 0$. Thus

$$\partial_x^\beta \Delta_k^t w = \partial_x^\beta \Delta_k^t u - \partial_x^\beta \Delta_k^t v \tag{4.18}$$

We first estimate $E(\partial_x^\beta \Delta_k^t w)^2$ where $k$ is any integer such that $k\delta_t \leq T_0$.

For this, independence of $W^j$, the mean square of the matringale term in \ref{eqn:4.18} is bounded by

$$E \left[ \partial_x^{\beta+1} w_{k\delta_t} \left( \frac{1}{N} \sum_{j=1}^N \Delta_k^j W^j \right) \right]^2 \leq C \delta_t^2 E(\partial_x^{\beta+1} w_{k\delta_t})^2 E \left( \frac{1}{N} \sum_{j=1}^N \Delta_k^j W^j \right)^2 \tag{4.19}$$

Next, for the mean square of the first term in \ref{eqn:4.18}

$$E \left( \partial_x^\beta (L_{u_{T_\wedge K\delta_t}} - L_{u_{T_\wedge K\delta_t}}(x)) \right)^2$$

$$\leq C E \left[ \left( \partial_x^{\beta+2} u_{T_\wedge K\delta_t} \right)^2 + \left( \partial_x^{\beta+2} v_{T_\wedge K\delta_t} \right)^2 \right] + C \sup_{x_0 \in T} C \max_{0 \leq k \leq \frac{T_0}{\beta}} \left( \| u_{T_\wedge K\delta_t} \|_{C^{1+\beta}} + \| v_{T_\wedge K\delta_t} \|_{C^{1+\beta}} \right).$$

Hence, using \ref{eqn:DeltakwTerm1} and \ref{eqn:DeltakwTerm2} we obtain

$$E \left( \partial_x^\beta (L_{u_{T_\wedge K\delta_t}} - L_{u_{T_\wedge K\delta_t}}) \Delta_k^t \right)^2 \leq C \delta_t^2. \tag{4.20}$$

By \ref{eqn:epsilon_Sob}, the mean square of the last term in \ref{eqn:4.18} is also bounded by $C \delta_t^2$. Thus, squaring \ref{eqn:4.18}, taking expected values and using Young’s inequality gives

$$E(\partial_x^\beta \Delta_k^t w)^2 \leq \frac{3 \delta_t}{N} E(\partial_x^{\beta+1} w_{k\delta_t})^2 + C \delta_t^2 + 3E(\partial_x^\beta \epsilon_k)^2. \tag{4.21}$$

Now for any $K \leq \frac{T_0}{\beta}$,

\begin{align*}
(\partial_x^\beta w_{K\delta_t})^2 &= (\partial_x^\beta u_{T_\wedge K\delta_t})^2 = 2 \sum_{k=0}^{K-1} \partial_x^\beta w_{k\delta_t} \partial_x^\beta \Delta_k^t w + \sum_{k=0}^{K-1} (\partial_x^\beta \Delta_k^t w)^2 \\
&= 2 \sum_{k=0}^{K-1} \partial_x^\beta w_{k\delta_t} \left( - \partial_x^\beta (L_{u_{T_\wedge K\delta_t}} - L_{u_{T_\wedge K\delta_t}}) \Delta_k^t w - \partial_x^{\beta+1} \Delta_k^t w \left( \frac{1}{N} \sum_{j=1}^N \Delta_k^j W^j \right) + \partial_x^\beta \epsilon_k \right) + \\
&\quad + \sum_{k=0}^{K-1} (\partial_x^\beta \Delta_k^t w)^2.
\end{align*}
Taking expected values, integrating in space using (4.21) and (4.16) gives

\[ E \| \partial_x^2 w_k \|_{L^2}^2 \leq -2\delta_t \sum_{k=0}^{K-1} E \int_{\Omega} \partial_x^2 w_k \partial_x^2 (u_{\tau\land k\delta_t} \partial_x u_{\tau\land k\delta_t} - v_{\tau\land k\delta_t} \partial_x v_{\tau\land k\delta_t}) \, dx + \]
\[ + 2 \sum_{k=0}^{K-1} E \int_{\Omega} \partial_x^2 w_k \partial_x^2 \varepsilon_k \, dx - \left(1 - \frac{1}{N}\right) \delta_t \sum_{k=0}^{K-1} E \int_{\Omega} (\partial_x^{2+1} w_{k\delta_t})^2 \, dx + C K \delta_t^2 \]

For the first term on the right of the inequality (4.22) note

\[ \partial_x^2 w_k \partial_x^2 (u_{\tau\land k\delta_t} \partial_x u_{\tau\land k\delta_t} - v_{\tau\land k\delta_t} \partial_x v_{\tau\land k\delta_t}) = \]
\[ \partial_x^3 w_k \partial_x^3 (w_k \partial_x u_{\tau\land k\delta_t} - v_{\tau\land k\delta_t} \partial_x w_k). \]

Observe that the mass (spatial mean) of solutions to equation (2.6) is constant in time. The same is true for solutions to (4.3). Thus, for all \( t \in [0, T] \), \( \int_u w \, dx = \int_0^T \int_{\Omega} u \, dx \), and hence \( \int_0^T w \, dt = 0 \). Thus integrating (4.23) in space and using the Poincaré inequality, the term involving \( u \) above is bounded by

\[ \left| \int_{\Omega} \partial_x^3 w_k \partial_x^3 (w_k \partial_x u_{\tau\land k\delta_t}) \, dx \right| \leq C \| \partial_x^2 w_k \delta_t \|_{L^2}^2 \| u_{\tau\land k\delta_t} \|_{C^3}. \]

For the term involving \( v \) in (4.23), when all the derivatives fall on \( w \) we have

\[ \partial_x^3 w_k \partial_x^3 (v_{\tau\land k\delta_t} \partial_x w_k) = \frac{1}{2} v_{\tau\land k\delta_t} \partial_x (\partial_x^2 w_k)^2, \]

and if we integrate by parts, we can avoid the extra derivative on \( w \). Thus

\[ \left| \int_{\Omega} \partial_x^3 w_k \partial_x^3 (v_{\tau\land k\delta_t} \partial_x w_k) \, dx \right| \leq C \| \partial_x^3 w_k \|_{L^2}^2 \| v_{\tau\land k\delta_t} \|_{C^3}. \]

Thus using (4.12) and the above estimates, the first term on the right of (4.22) is bounded by

\[ -2\delta_t \int_{\Omega} \partial_x^3 w_k \partial_x^3 (u_{\tau\land k\delta_t} \partial_x u_{\tau\land k\delta_t} - v_{\tau\land k\delta_t} \partial_x v_{\tau\land k\delta_t}) \, dx \leq C \delta_t E \| \partial_x^2 w_k \|_{L^2}^2. \]

For the second term in (4.22), we know \( w_k \delta_t \) is \( \mathcal{F}_{k\delta_t} \)-measurable. Thus using (4.17) and the Cauchy-Schwartz inequality we obtain

\[ E \| \partial_x^2 w_k \|_{L^2}^2 \leq C \delta_t^{1/2} + C \sum_{k=0}^{K-1} E \| \partial_x^2 w_k \|_{L^2}^2 \delta_t. \]
The remainder of the proof is an elementary discrete Gronwall argument. Let
\[ y_K = C\delta_t^{1/2} + C \sum_{k=0}^{K-1} E \| \partial_x^2 w_{k\delta_t} \|_{L^2} \delta_t. \]
Then
\[ y_{k+1} - y_k = C\delta_t E \| \partial_x^2 w_{k\delta_t} \|_{L^2} \leq C\delta_t y_k \]
and hence
\[ y_{k+1} \leq (1 + C\delta_t)^k y_k. \]
Iterating, and using \( y_0 = C\delta_t^{1/2} \) gives
\[ y_k \leq (1 + C\delta_t)^k C\delta_t^{1/2}. \]
Since \( k \leq \frac{T}{\delta_t} \) this gives
\[ \max_{k \leq \frac{T}{\delta_t}} y_k \leq C\delta_t^{1/2} \max_{\delta_t > 0} (1 + C\delta_t)^T \delta_t^{1/2} \leq C\delta_t^{1/2} e^{CT_0}. \]
This proves (2.9) for all times \( t \) which are an integer multiple of \( \delta_t \). Since for any \( x \in T \), and \( k \leq \frac{T}{\delta_t} \) we elementarily have
\[ \sup_{k \delta_t \leq t \leq (k+1) \delta_t} E \| \partial_x^3 v_{\tau \wedge t}(x) - \partial_x^3 v_{\tau \wedge k \delta_t}(x) \|^2 \leq C\delta_t \]
and
\[ \sup_{k \delta_t \leq t \leq (k+1) \delta_t} E \| \partial_x^3 u_{\tau \wedge t}(x) - \partial_x^3 u_{\tau \wedge k \delta_t}(x) \|^2 \leq C\delta_t \]
completing the proof.

5. Proof of Lemma 2.6

In this section we establish uniform in time bounds for the solution of (2.6) and prove as in Lemma 2.6. We do this via the following two Lemmas:

Lemma 5.1. Let \( u_0 \in C^\infty(T), \ T > 0, \) and suppose \( v \in C^\infty([0, T]; T) \) is a solution to (2.6) with initial data \( u_0 \) and periodic boundary conditions. Then for any \( s \in \mathbb{Z}^+ \), there exists a constant \( V_s = V_s(s, T, \|u_0\|_{H^s}) \) such that
\[ \sup_{0 \leq t \leq T} \| v_t \|_{H^s} \leq V_s \]
almost surely.

Lemma 5.2. Let \( u_0 \in C^\infty(T) \), and suppose \( v \in C^\infty([0, \infty), T) \) is a solution to (2.6) with initial data \( u_0 \) and periodic boundary conditions. Then for any \( s \in \mathbb{Z}^+, T > 0, \) there exists a constant \( V_s = V_s(s, T, \|u_0\|_{L^2}) \) such that
\[ \sup_{t \geq T} \| v_t \|_{H^s} \leq V_s \]
almost surely.

We draw attention to the fact that apriori bounds are almost sure! Indeed, applying Itô’s formula to \( \| v_t \|^2_{L^2} \) immediately yields an equation with no martingale part (see equation (5.2) below).

Given Lemmas 5.1 and 5.2 the proof of 2.6 is now immediate.
Proof of Lemma 2.6. Given the almost sure apriori bounds in Lemmas 5.1 and 5.2, existence of solutions to (2.6) follows via standard methods. The time global bound (2.10) is also an immediate consequence of Lemmas 5.1 and 5.2.

We devote the remainder of this section to proving Lemmas 5.1 and 5.2.

Proof of Lemma 5.1. We prove Lemma 5.1 via energy estimates. First note that Itô’s formula and (2.6) give

\[
\begin{align*}
\frac{d}{dt} \|v_t\|_{L^2}^2 &= 2v_t \cdot \nabla v_t + \frac{1}{N^2} \sum_{j=1}^{N} (\partial_x v_t)^2 dt \\
&= -2v_t^2 \partial_x v_t dt + v_t \partial_x^2 v_t dt - 2\frac{v_t \partial_x v_t}{N} \sum_{j=1}^{N} dW_j + \frac{1}{N} (\partial_x v_t)^2 dt.
\end{align*}
\]

Integrating in space, and using \( \int_0^T v_t \partial_x v_t dx = 0 = \int_0^T v_t^2 \partial_x v_t dx \) gives

\[
(5.2) \quad \partial_t \|v_t\|_{L^2}^2 = - \left( 1 - \frac{1}{N} \right) \|\partial_x v_t\|_{L^2}^2 \quad \Rightarrow \quad \|v_t\|_{L^2} \leq C \|v_0\|_{L^2}
\]

almost surely.

A similar calculation shows \( \|v_t\|_{L^p} \leq \|v_0\|_{L^p} \) for all \( p \geq 2 \), and hence \( \|v_t\|_{L^\infty} \leq \|v_0\|_{L^\infty} \). Recall \( s \geq 1 \) by assumption, and so the Sobolev embedding theorem shows \( \|v_0\|_{L^\infty} \leq C \|u_0\|_{H^s} \) for some absolute constant \( C \).

Now, differentiating (2.6) with respect to \( x \) and applying Itô’s formula to \( (\partial_x v_t)^2 \) we obtain

\[
\begin{align*}
\frac{d}{dt} (\partial_x v_t)^2 &= 2\partial_x v_t \cdot \nabla (\partial_x v_t) + \frac{1}{N} \|\partial_x^2 v_t\|^2 dt \\
&= -2\partial_x v_t \left( \partial_x (v_t \partial_x v_t) dt - \frac{1}{2} \partial_x^3 v_t dt + \frac{\partial_x^2 v_t}{N} \sum_{j=1}^{N} dW_j \right) + \frac{1}{N} \|\partial_x^2 v_t\|^2 dt
\end{align*}
\]

Integrating with respect to \( x \) on \([0, 1]\), and noting that \( \int_0^1 \partial_x v_t \partial_x^2 v_t dx = 0 \), gives

\[
\begin{align*}
\frac{d}{dt} \|\partial_x v_t\|_{L^2}^2 &= - \left( 1 - \frac{1}{N} \right) \|\partial_x^2 v_t\|_{L^2}^2 dt + \frac{1}{2} \int_0^1 \partial_x^2 v_t (v_t \partial_x v_t) dx dt \\
&\Rightarrow \partial_t \|\partial_x v_t\|_{L^2}^2 \leq - \frac{1}{4} \|\partial_x^2 v_t\|_{L^2}^2 + 8 \|v_t\|_{L^\infty}^2 \|\partial_x v_t\|_{L^2}^2 \\
&\leq - \frac{1}{4} \|\partial_x^2 v_t\|_{L^2}^2 + 8 \|u_0\|_{L^\infty}^2 \|\partial_x v_t\|_{L^2}^2,
\end{align*}
\]

almost surely. Thus, (5.2), (5.3) and Gronwall’s inequality gives

\[
(5.4) \quad \|v_t\|_{H^1} \leq C_1 e^{c_0 t} \quad \text{and} \quad \int_0^t \|v_t\|_{H^2}^2 dt' \leq C_1 e^{c_0 t},
\]

almost surely, for some constants \( C_1 = C_1(\|u_0\|_{H^1}) \) and \( c_0 = c_0(\|u_0\|_{L^\infty}) \).

For the remainder of this proof we adopt the convention that \( c, C \) denote absolute constants, \( C_s = C_s(s, \|u_0\|_{H^s}) \) denotes a constant depending only on \( s, \|u_0\|_{H^s} \) and

\[^9\text{This can alternately be shown using a version of the maximum principle} \[16\].\]
$c_0$ denotes a constant depending only on $\|u_0\|_{L^\infty}$. The exact value of these constants are immaterial, and we will allow them to change from line to line.

Similar to (5.3), differentiating (2.6) twice with respect to $x$, applying Itô's formula to $(\partial_x^2 v_t)^2$, integrating in space, noting $\int_x \partial_x^2 v_t \partial_x^2 v_t \, dx = 0$ and using Hölder's inequality gives

$$\partial_t \|\partial_x^2 v_t\|^2_{L^2} \leq - \left(1 - \frac{1}{N}\right) \|\partial_x^2 v_t\|^2_{L^2} + 2 \|\partial_x^2 v_t\|_{L^2} \|\partial_x (v_t \partial_x v_t)\|_{L^2},$$

$$\leq - c \|\partial_x^2 v_t\|^2_{L^2} + C \left(\|\partial_x v_t\|^2_{L^\infty} + \|v_t\|^2_{L^\infty}\right) \|\partial_x^2 v_t\|^2_{L^2},$$

almost surely, where the last inequality is obtained by the Sobolev embedding theorem. Using (5.4), this gives

$$\|v_t\|_{H^2} \leq C e^{\int_0^t \|v_t\|_{H^2}^2 \, dt} \leq C e^{C_1 e^{c_0 t}} \quad \text{and} \quad \int_0^t \|v_t\|^2_{H^2} \, dt' \leq C e^{C_1 e^{c_0 t}},$$

almost surely. Proceeding inductively, suppose we know

$$\|v_t\|_{H^{s+1}} \leq C_s \exp(C_{s-1} \exp(C_{s-2} \ldots \exp(C_0 t) \ldots))$$

(5.5)

holds almost surely for some $s \in \mathbb{Z}^+$. Differentiating (2.6) $s+1$ times with respect to $x$, applying Itô's formula for $(\partial_x^{s+1} v_t)^2$, and integrating in space we obtain

$$d \|\partial_x^{s+1} v_t\|^2_{L^2} = - \left(1 - \frac{1}{N}\right) \|\partial_x^{s+1} v_t\|^2_{L^2} + 2 \|\partial_x^{s+2} v_t\|_{L^2} \|\partial_x (v_t \partial_x v_t)\|_{L^2} \, dt,$$

since $\int_x \partial_x^{s+1} v_t \partial_x^{s+2} v_t \, dx = 0$. Thus

$$\partial_t \|\partial_x^{s+1} v_t\|^2_{L^2} \leq - c \|\partial_x^{s+2} v_t\|^2_{L^2} + C \left(\|\partial_x v_t\|^2_{L^\infty} + \|\partial_x v_t\|^2_{L^\infty} + \cdots + \|\partial_x^{s+1} v_t\|^2_{L^\infty}\right),$$

$$\leq - c \|\partial_x^{s+2} v_t\|^2_{L^2} + C \left(\|\partial_x v_t\|^2_{L^\infty} + \cdots + \|\partial_x v_t\|^2_{L^\infty} + \cdots + \|\partial_x^{s+1} v_t\|^2_{L^\infty}\right),$$

$$\leq - c \|\partial_x^{s+2} v_t\|^2_{L^2} + C \|v_t\|^2_{H^{s+1}} \|\partial_x^{s+1} v_t\|^2_{L^2},$$

almost surely. Thus by Gronwall's Lemma

$$\|v_t\|_{H^{s+1}} \leq C_{s+1} \exp\left(\int_0^t \|v_t\|^2_{H^{s+1}} \, dt'\right) \leq C_{s+1} \exp(C_s \exp(C_{s-1} \ldots \exp(C_0 t) \ldots))$$

almost surely. Further

$$\int_0^t \|v_t\|^2_{H^{s+1}} \, dt' \leq C_{s+1} \exp(C_s \exp(C_{s-1} \ldots \exp(C_0 t) \ldots)),$$

almost surely, completing the inductive step. By induction, (5.5) holds for all $s \in \mathbb{Z}^+$ completing the proof. \qed
Proof of Lemma 5.2. We prove Lemma 5.2 via a bootstrapping argument in Fourier space. To fix notation, for $n \in \mathbb{Z}$, we use $\hat{f}(n) = \int e^{-2\pi inx} f(x) \, dx$ to denote the $n$th Fourier coefficient of $f$.

On Fourier coefficients, using $u \partial_x u = \frac{1}{2} \partial_x u^2$, equation (2.6) reduces to

$$
\hat{d} \hat{v}(n) + \frac{2\pi in}{N} \hat{v}(n) \sum_{j=1}^{N} dW^j_t + 2\pi^2 n^2 \hat{v}(n) \, dt + \pi in \sum_{m \in \mathbb{Z}} \hat{v}(n-m) \hat{v}(m) \, dt = 0
$$

for every $n \in \mathbb{Z}$.

By Itô’s formula applied to (5.6)

$$
(5.7) \quad d |\hat{v}(n)|^2 = \frac{\hat{v}(n)}{N} \hat{d} \hat{v}(n) \, dt + \hat{v}(n) \hat{d} \hat{v}(n) + \frac{4\pi^2 n^2}{N} |\hat{v}(n)|^2 \, dt
$$

where $\overline{\hat{v}(n)}$ denotes the complex conjugate of $\hat{v}(n)$, and

$$
\hat{B}(n) = \sum_{m \in \mathbb{Z}} \hat{v}(n-m) \hat{v}(m),
$$

is the non-linear Fourier coupling in (5.6). Using $N > 1$ and Young’s inequality in (5.7) gives

$$
(5.8) \quad \frac{1}{n^2} |\hat{v}(n)|^2 \leq -2\pi^2 n^2 |\hat{v}(n)|^2 + 2\pi n |\hat{v}(n)| |\hat{B}(n)|
$$

almost surely, where, as before $c, C$ are absolute constants (independent of $u_0, T$), which may change from line to line. Thus, for any $t' \geq 0$ we have

$$
(5.9) \quad |\hat{v}(n)|^2 \leq |\hat{v}_{t'_0'}(n)|^2 e^{-\frac{\pi^2 n^2}{2} t'} + C \int_{t'_0}^{t'} e^{-c n^2 (t'-t')} |\hat{B}(n)|^2 \, dt'
$$

almost surely, by Gronwall’s inequality.

By Parseval’s identity we know $|\hat{B}(n)| \leq \|\hat{v}(n)\|_{L^2}$, and by conservation of energy (equation (5.2)) this gives $|\hat{B}(n)| \leq \|u_0\|_{L^2}$ almost surely. Thus the second term in the previous inequality is bounded from above by $\frac{C}{\pi n^2} \|\hat{u}(n)\|_{L^2}$. Since $|\hat{u}_{t'_0'}|^2 \leq \|\hat{u}_{t'_0'}\|_{L^2}^2 \leq \|u_0\|_{L^2}^2$, given a lower bound on $t - t'_0$, we can certainly arrange the same inequality for the first term. Thus choosing $t_1 = \frac{T}{2}$ for instance, and applying (5.9) with $t'_0 = 0$, we obtain

$$
(5.10) \quad \sup_{t \geq t_1} |\hat{v}(n)|^2 \leq \frac{C_0}{n^2}
$$

almost surely, where $C_0 = C_0(\|u_0\|_{L^2}, T)$ is some constant.

Now we bootstrap, and use (5.10) to obtain a better estimate on $\hat{B}_t$. Assume inductively that for some $\alpha \in \mathbb{Z}^+$, and $t_\alpha = \frac{\alpha}{\alpha + 1} T$, we have

$$
(5.11) \quad \sup_{t \geq t_\alpha} |\hat{v}(n)|^2 \leq \frac{C_\alpha}{n^{\alpha + 1}}
$$

almost surely. Here $C_\alpha = C_\alpha(\|u_0\|_{L^2}, T, \alpha)$ is a constant which we allow to change from line to line if necessary. We will now establish (5.11) for $\alpha + 1$. Note that
almost surely, for any \( t > t_\alpha \), we have

\[
|B_t(n)| \leq \sum_{m \in \mathbb{Z}} |\hat{v}_t(n - m)| |\hat{v}_t(m)| \leq 2 \sum_{|m| \geq |n|/2} |\hat{v}_t(n - m)| |\hat{v}_t(m)|
\]

\[
\leq 2 \|v_t\|_2 \left( \sum_{|m| \geq |n|/2} |\hat{v}_t(m)|^2 \right)^{1/2} \leq 2 \|u_0\|_2 \left( \sum_{|m| \geq |n|/2} \frac{C_\alpha}{m^{\alpha+1}} \right)^{1/2}
\]

(5.12)

Now returning to (5.9) and choosing \( t'_0 = t_\alpha \), we see that the second term is bounded by \( \frac{C}{C_\alpha} \frac{C_\alpha}{m^{\alpha+1}} = C C_\alpha \). For any \( t > t_\alpha + 1 \), we can certainly arrange the same inequality for the first term, and hence this establishes (5.11) for \( \alpha + 1 \).

Finally note that if (5.11) holds for \( \alpha \), then (5.1) holds for any \( s < \frac{\alpha}{\alpha+1} \), finishing the proof.

\[ \square \]

6. Proof of Lemma 2.7

In this section, we prove the almost sure \( C^n(\mathbb{T}) \) bounds on \( u \) stated in Lemma 2.7. We need a few preliminary results first.

**Proposition 6.1** (Local existence without resetting). Let \( u_{t_0} \) be a \( C^1(\mathbb{T}) \) valued \( \mathcal{F}_{t_0} \)-measurable random variable such that

\[
\|u_{t_0}\|_{C^1} \leq U_1^0
\]

almost surely. There exists \( T_0 = T_0(U_1^0) \), independent of \( N \), such that the solution to (2.1) exists on the interval \([t_0, t_0 + T_0]\). Further if for some \( n \geq 1 \), \( u_{t_0} \) is a \( C^n(\mathbb{T}) \) valued, \( \mathcal{F}_{t_0} \)-measurable random variable with

\[
\|u_{t_0}\|_{C^n} \leq U_n^0
\]

almost surely, then there exists \( U_n = U_n(U_n^0, n) \) such that

\[
(6.1) \quad \sup_{t_0 \leq t \leq t_0 + T_0} \|u_t\|_{C^n} \leq U_n
\]

almost surely.

Proposition 6.1 can be proved using a standard Picard’s iteration. A proof of the analogous result for the Navier-Stokes equations appeared in the appendix of [11] (see also [9,10]). The proof of 6.1 is very similar, and we do not provide it here.

**Lemma 6.2.** Let \( I : \mathbb{R} \rightarrow \mathbb{R} \) denote the identity function, \( d \in [0,1) \), and let \( \lambda \in C^n(\mathbb{T}) \) be a periodic function such that \( \|\partial_x \lambda\|_{L^\infty} \leq d \). Then there exists a constant \( c_{n-1} = c_{n-1}(\|\partial_x^{n-1} \lambda\|_{L^\infty}, d, n) \) such that for any \( f \in C^0(\mathbb{R}) \),

\[
(6.2) \quad \|\partial_x^n f \circ (I + \lambda)\|_{L^\infty} \leq \|\partial_x^n f\|_{L^\infty} \left( 1 + \|\partial_x \lambda\|_{L^\infty} \right)^n + c_{n-1} \|\partial_x^n \lambda\|_{L^\infty}
\]

\[
(6.3) \quad \|\partial_x^n (I + \lambda)^{-1}\|_{L^\infty} \leq c_{n-1} \|\partial_x^n \lambda\|_{L^\infty}
\]

for \( n > 1 \).

**Remark.** Note that since \( \|\partial_x \lambda\|_{L^\infty} < 1 \), the function \( I + \lambda \) is a \( C^1(\mathbb{R}) \) diffeomorphism of \( \mathbb{R} \). The notation \((I + \lambda)^{-1}\) in (6.3) refers to the inverse of the \( C^1(\mathbb{R}) \) diffeomorphism \( I + \lambda \).
Lemma 6.3. If \( \lambda = I \) the identity map, \( U \) surely. Let \( u \subset C \) not affect our proof below. Our first step is to obtain almost sure the convention that \( c \) on the Eulerian and Lagrangian displacements. Throughout this section, we use a constant.

Before. This proves (6.3).

Now for any two \( f, g \in C^n(\mathbb{R}) \), we have

\[
\partial_x^n (f \circ g) = \sum_{m=1}^{n} (\partial_x^m f) \circ g \sum_{k_1 + \ldots + k_m = n \atop k_i \geq 1} \prod_{i=1}^{m} \partial_x^{k_i} g.
\]

To prove (6.2), we set \( g = I + \lambda \). The term in (6.4) corresponding to \( m = n \) gives the first term of (6.2). When \( m < n \), we notice that \( k_i > 1 \) for at least one \( i \), and \( k_j \leq n - 1 \) for all other \( j \). Thus \( \|\partial_x^j (I + \lambda)\|_{L^\infty} = \|\partial_x^j \lambda\|_{L^\infty} \leq c(n)\|\partial_x^j \lambda\|_{L^\infty} \). The remaining terms \( \partial_x^j (I + \lambda) \), \( j \neq i \) in the product can be bounded by \( c_{n-1} \). This proves (6.2).

For (6.3), set \( X = I + \lambda \) and \( A = X^{-1} \). Since \( n > 1 \), \( \partial_x^n (A \circ X) \equiv 0 \), and using (6.4) we obtain

\[
\partial_x^n A \big|_X = \frac{-1}{(\partial_x X)^n} \sum_{m=1}^{n-1} \partial_x^m A \big|_X \sum_{k_1 + \ldots + k_m = n \atop k_i \geq 1} \prod_{i=1}^{m} \partial_x^{k_i} A.
\]

By induction, one can assume that \( \|\partial_x^m A\|_{L^\infty} \leq c_{n-1} \) for all \( m \leq n - 1 \). Since \( d < 1 \), \( \|\partial_x A\|_{L^\infty} \leq \frac{1}{1-d} \), and remaining terms can be bounded by the same argument as before. This proves (6.3). \( \square \)

Lemma 6.3. Let \( n \in \mathbb{N} \), \( u_{t_0} \) be a bounded, \( C^n(\mathbb{T}) \) valued, \( \mathcal{F}_{t_0} \)-measurable random variable. For \( k \in \{0, \ldots, k\} \), let \( k \in \mathbb{N} \) be a constant such that \( \|u_{t_0}\|_{C^k} \leq U^0_k \) almost surely. Let \( u \) be the solution of (2.1) with initial data \( u_t = u_{t_0} \) when \( t = t_0 \). If \( n > 1 \), there exists \( \Omega' \in \mathcal{F}_{t_0} \) with \( P(\Omega') = 1 \), \( T_0 = T_0(U^0_0) > t_0 \) and a constant \( c_{n-1} = c_{n-1}(U^0_{n-1}, n) \) such that

\[
\|\partial_x^n u_t(\omega')\|_{L^\infty} \leq U^0_k (1 + c_{n-1}(t - t_0))
\]

for all \( \omega' \in \Omega' \), \( t \in [t_0, t_0 + T_0] \). For \( n = 1 \), (6.5) holds with \( c_0 \) to be an absolute constant.

Proof. For simplicity, we assume \( t_0 = 0 \). One can check that this assumption does not affect our proof below. Our first step is to obtain almost sure \( C^1(\mathbb{T}) \) estimates on the Eulerian and Lagrangian displacements. Throughout this section, we use the convention that \( c_{n-1} = c_{n-1}(U^0_{n-1}, n) \) is a constant depending only on \( n \) and \( U^0_{n-1} \) (or an absolute constant for \( n = 1 \)), which can change from line to line.

Let \( T_0 = T_0(U^0_0) \) be the local existence time given by Proposition 6.1, and \( c_1 = c_1(U^0_1) \) the almost sure bound on \( \|u_t\|_{C^1} \) from (6.1). Let \( I : \mathbb{R} \to \mathbb{R} \) be the identity map, \( X' \), \( A' \) respectively be as in (2.1), (2.2), with \( \tau = T_0 \). Define \( \lambda_0 = X' - I \), \( \ell_0 = A' - I \).

Differentiating (2.1) with respect to \( x \) we obtain

\[
\|\partial_x \lambda_0\|_{L^\infty} \leq \int_0^t \|\partial_x u_s\|_{L^\infty} (1 + \|\partial_x \lambda_s\|_{L^\infty})
\]
almost surely, for \( t \in [0, T_0] \). By Gronwall’s lemma,
\[
\| \partial_x \lambda^i \|_{L^\infty} \leq c e^{c t} \int_0^t \| \partial_x u_s \|_{L^\infty} \ ds \quad \text{a.s.}
\]
for \( t \in [0, T_0] \). Recall \( t \leq T_0 \), \( c_1 \) only depends on \( U_0^1 \), and for all \( s \leq T_0 \), \( \| \partial_x u_s \|_{L^\infty} \leq U_0^1 \) almost surely. Thus, as \( T_0 \) is allowed to depend on \( U_0^1 \), by making \( T_0 \) smaller if necessary we can arrange

\[
\| \partial_x \lambda^i \|_{L^\infty} \leq c_0 \int_0^t \| \partial_x u_s \|_{L^\infty} \ ds \quad \text{and} \quad \sup_{0 \leq t \leq T_0} \| \partial_x \lambda^i \|_{L^\infty} \leq \frac{1}{2}
\]
almost surely, for some absolute constant \( c_0 \). Now
\[
\partial_x \ell^i_t = \partial_x A^i_t - 1 = \frac{1}{(\partial_x \lambda^i_t) \circ A^i_t} - 1 = - \frac{(\partial_x \lambda^i_t) \circ A^i_t}{1 + (\partial_x \lambda^i_t) \circ A^i_t}
\]
amost surely. Thus we must have

\[
\| \partial_x \ell^i_t \|_{L^\infty} \leq 2 \| \partial_x \lambda^i_t \|_{L^\infty}
\]
almost surely for \( t \in [0, T_0] \). Using (2.3) and (6.7) we have
\[
\| \partial_x u_t \|_{L^\infty} \leq \frac{1}{N} \sum_{i=1}^N \| \partial_x u_0 \|_{L^\infty} \left( 1 + \| \partial_x \ell^i_t \|_{L^\infty} \right)
\]
\[
\leq \frac{1}{N} \sum_{i=1}^N \| \partial_x u_0 \|_{L^\infty} \left( 1 + 2 \| \partial_x \lambda^i_t \|_{L^\infty} \right)
\]
\[
\leq \| \partial_x u_0 \|_{L^\infty} + 2c_0 \int_0^t \| \partial_x u_s \|_{L^\infty} \ ds
\]
amost surely for \( t \in [0, T_0] \). This proves (6.5) for \( n = 1 \).

For \( n > 1 \), local existence (Proposition 6.1) guarantees that \( \| u_t \|_{C^{n-1}} \leq c_{n-1} \) almost surely for \( t \in [0, T_0] \), where \( c_{n-1} = c_{n-1}(U_{n-1}^0, n) \). Assume by induction that the bound (6.5) holds for some integer \( n - 1 \). This bound and equation (2.1) immediately imply that \( \| \partial_x \lambda^i_t \|_{C^{n-2}} \leq c_{n-1} \) almost surely for \( t \in [0, T_0] \). Equations (6.6) and (6.3) will imply \( \| \partial_x \ell^i_t \|_{C^{n-2}} \leq c_{n-1} \) almost surely for \( t \in [0, T_0] \).

Thus using equations (2.1) and (6.2) we obtain
\[
\| \partial_x \lambda^i_t \|_{L^\infty} \leq \int_0^t \| \partial_x^2 [u_s \circ (I + \lambda^i_s)] \|_{L^\infty} \ ds
\]
\[
\leq c_{n-1} \int_0^t \| \partial_x^2 u_s \|_{L^\infty} + \| \partial_x \lambda^i_s \|_{L^\infty} \ ds
\]
amost surely. Using Gronwall’s lemma this implies

\[
\| \partial_x \lambda^i_t \|_{L^\infty} \leq c_{n-1} \int_0^t \| \partial_x^2 u_s \|_{L^\infty} \ ds
\]

\[\text{We remark that our somewhat unusual notation } \| \partial_x \lambda^i_t \|_{C^{n-2}} \text{ instead of } \| \lambda_t^i \|_{C^{n-1}} \text{ is necessary. This is because it is impossible to obtain almost sure bounds on } \| \lambda^i_t \|_{L^\infty}. \text{ However, as our argument shows, we can obtain almost sure bounds on } \| \partial_x \lambda^i \|_{L^\infty} \text{ for any } k \geq 1.\]
almost surely. Here we absorbed the constant $e^{c_{n-1}t}$ into $c_{n-1}$, which is valid as $t \leq T_0 = T_0(U_0^1)$. Now
\[
\|\partial_x^n u_t\|_{L^\infty} \leq \frac{1}{N} \sum_{i=1}^{N} \left(\|\partial_x^n u_0\|_{L^\infty} \left(1 + \|\partial_x^i u_t\|_{L^\infty}\right)^n + c_{n-1} \|\partial_x^i u_t\|_{L^\infty}\right)
\]
\[
\leq \frac{1}{N} \sum_{i=1}^{N} \left(\|\partial_x^n u_0\|_{L^\infty} \left(1 + 2 \|\partial_x^i u_t\|_{L^\infty}\right)^n + c_{n-1} \|\partial_x^i u_t\|_{L^\infty}\right)
\]
\[
\leq \|\partial_x^n u_0\|_{L^\infty} \left(1 + c_{n-1}t\right) + c_{n-1} \int_0^t \|\partial_x^n u_s\|_{L^\infty} \, ds
\]
almost surely, where we used (6.3) and (6.7) to obtain the second inequality, and equations (6.6) and (6.8) to obtain the third inequality. Now Gronwall’s Lemma gives (6.5), where we again absorb the exponential factor $e^{c_{n-1}t}$ into $(1 + c_{n-1}t)$, by replacing $c_{n-1}$ with a larger constant, which by our convention we still denote by $c_{n-1}$.

**Proof of Lemma 2.7.** By Proposition 6.1 existence will follow if we establish (2.11) for $n = 1$. We prove (2.11) by induction. Since the constant $c_0$ in Lemma 6.3 is absolute, the proof for $n = 1$ is identical to the proof of the inductive step. Thus we only prove the inductive step.

Assume that (2.11) holds for $n-1$, choose $c_{n-1} = c_{n-1}(U_{n-1})$ to be the constant from Lemma 6.3. Thus whenever $\delta_t < T_0$, (6.9)
\[
\|\partial_x^n u^\delta_{(k+1)\delta_t}\|_{L^\infty} \leq (1 + c_{n-1}\delta_t) \|\partial_x^n u^\delta_{k\delta_t}\|_{L^\infty} \quad \text{a.s.}
\]
holds for all $k \leq T_0/\delta_t$. Iterating this we have
\[
\|\partial_x^n u^\delta_t\|_{L^\infty} \leq (1 + c_{n-1}\delta_t) T_0/\delta_t \|\partial_x^n u_0\|_{L^\infty} \quad \text{a.s.}
\]
for all $t \leq T_0$. Thus we choose $U_n$ to be given by
\[
U_n = \|\partial_x^n u_0\|_{L^\infty} \sup_{\delta > 0} (1 + c_{n-1}\delta)^{T_0/\delta}.
\]
From (4.3) we see that $\int x u^\delta_t$ is conserved almost surely. Since $u^\delta_t$ is periodic, a bound on $\|\partial_x^n u^\delta_t\|_{L^\infty}$ will give us a bound on $\|u^\delta_t\|_{C^\infty}$, concluding the proof. □

**7. Proof of Proposition 2.8.** In this section we prove Proposition 2.8. We reintroduce an $N$ as a superscript to explicitly keep track of the dependence of our processes on $N$, and prove convergence as $N \to \infty$.

**Proof of Proposition 2.8.** Let $u^n_t = v^n_t - u^b_t$. Then (1.1) and (2.6) give
(7.1) \[
dw^n_t + w^n_t \partial_x v^n_t dt + u^n_t \partial_x w^n_t dt - \frac{1}{2} \partial_x^2 w^n_t dt + \frac{\partial_x v^n_t}{N} \sum_{i=1}^{N} dw^j_i = 0.
\]
Thus, by Itô’s formula
\[
\frac{1}{2} \, \|w^n_t\|_{L^2}^2 + \left(\int_T (w^n_t)^2 \partial_x v^n_t dx\right) dt + \left(\int_T u^n_t w^n_t \partial_x w^n_t dx\right) dt + \frac{1}{2} \, \|\partial_x w^n_t\|_{L^2}^2 dt +
\]

\[
\left( \int_T w_t^N \partial_x v_t^N \, dx \right) \left( \frac{1}{N} \sum_{j=1}^N dW_t^j \right) \, dt = \frac{1}{2N} \| \partial_x v_t^N \|_{L^2}^2 \, dt.
\]

Taking expectations and integrating by parts we obtain
\[
\partial_t E \| w_t^N \|_{L^2}^2 + E \left[ \int_T (w_t^N)^2 \left( 2 \partial_x v_t^N - \partial_x u_t^N \right) \, dx \right] + E \| \partial_x w_t^N \|_{L^2}^2 = \frac{1}{N} E \| \partial_x v_t^N \|_{L^2}^2.
\]

By Lemma 2.6 and the Sobolev embedding theorem, there exists a constant \( C = C(s, \| u_0 \|_{H^s}) \), independent of \( N \), such that
\[
\sup_{t \geq 0} \| \partial_x v_t \|_{L^\infty} \leq C
\]
almost surely. It is well known that the same estimate holds for \( \partial_x u_t^N \). Further, since \( E \| \partial_x v_t \|_{L^2}^2 \leq \sup_t \| \partial_x v_t \|_{L^\infty}^2 \), making \( C \) larger if necessary we have
\[
\sup_{t \geq 0} E \| \partial_x v_t \|_{L^2}^2 \leq C.
\]

Thus
\[
\partial_t E \| w_t^N \|_{L^2}^2 \leq V \left( E \| w_t^N \|_{L^2}^2 + \frac{1}{N} \right).
\]

and, since \( w_0 = 0 \), Gronwall’s lemma gives
\[
E \| w_t^N \|_{L^2}^2 \leq \frac{1}{CN} \left( e^{Ct} - 1 \right)
\]
finishing the proof. 

\[\square\]

References


THE REGULARIZING EFFECTS OF RESETTING


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