BLOW-UP OF SOLUTIONS TO A $p$-LAPLACE EQUATION

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1. Introduction. Our work is motivated by the issue of material failure initiation. Failure initiation occurs in zones of high concentrations of extreme electric or current fields, heat fluxes, and mechanical loads. Such zones are normally created by external loads amplified by composite microstructure. Therefore, the main focus of this study is on high contrast concentrated composites in which conditions for high field concentration have been created. A $p$-Laplace equation on domains with two spherical inclusions is a prototype setup for models of two-phase nonlinear composite materials.

We assume that the composite occupies a bounded domain in $\mathbb{R}^d$, $d = 2, 3$. We consider a composite material that consists of a background medium that contains two particles of a different material. The distance $\delta$ between particles is much smaller than their sizes. We also assume that particles are perfectly conducting and the background medium is described by the current-electric field relation

$$J = \sigma |E|^{p-2}E, \quad p > 2, \quad p \in \mathbb{N}. \quad (1)$$

Relation (1) describes various physical phenomena which include deformation theory of plasticity (e.g., [12, 22, 24]), where $E$ and $J$ are identified with infinitesimal strain and stress, respectively, nonlinear dielectrics (e.g., [8, 11, 16, 25]), where $E$ and $J$ are identified with electric field and current, respectively, and fluid flow (e.g., [1, 23]), where $E$ and $J$ are identified with rate of strain and fluid stress, respectively. For definiteness we say that $u$ is the electric potential, and the electric field $E = \nabla u$.

The energy in the thin gaps between neighboring particles of the composite described by $\int J \cdot E$ exhibits singular behavior as $\delta \to 0$ [14], as well electric field $\nabla u$ in the composite. If $p = 2$, then [5] (see also [20, 27, 28]) typically there exists $C > 0$
independent of $\delta$ such that
\begin{equation}
\frac{1}{C\sqrt{\delta}} \leq \|\nabla u\|_{L^\infty} \leq \frac{C}{\sqrt{\delta}} \quad \text{for} \quad d = 2, \quad \frac{\log \delta^{-1} C}{C \delta} \leq \|\nabla u\|_{L^\infty} \leq \frac{C \log \delta^{-1}}{\delta} \quad \text{for} \quad d = 3.
\end{equation}

The main result of this paper is the first asymptotic estimate for any $p > 2$. Namely, we show that typically
\begin{equation}
\lim_{\delta \to 0} \|\nabla u\|_{L^\infty} \delta^\gamma = C, \quad C > 0, \quad \text{for} \quad d = 2, 3, p > 2,
\end{equation}

where the constant $\gamma = \gamma(p, d) > 0$ and $C$ is explicitly computable. In the linear case the blow-up of the electric field is stronger than that in the nonlinear case (see Table 1).

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<th>$d$</th>
<th>$p = 2$</th>
<th>$p &gt; 2$</th>
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<tr>
<td>2</td>
<td>$\delta^{-\frac{1}{2}}$</td>
<td>$\delta^{-\frac{1}{p-1}}$</td>
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<tr>
<td>3</td>
<td>$\delta^{-\frac{1}{2}} (\delta \log \delta)^{-1}$</td>
<td>$\delta^{-\frac{1}{p-1}}$</td>
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We use the method of sub- and supersolutions to obtain (3). This method is applicable when $p = 2$ as well. In this case our argument is shorter than those in [5, 20, 27, 28]. Further, it allows us to compute the precise limit (3). This is the main contribution of our work. We prove (3) for physically relevant dimensions $d = 2, 3$ and for only two inclusions of circular shape. Similarly we may obtain estimates when $d > 3$. Further, our methods can be applied to arbitrary inclusions with smooth boundaries. In this latter case, the estimates depend on the curvatures of the boundaries of the inclusions.

Fluxes in high contrast densely packed particulate composites typically develop singularities in thin gaps between adjacent particles. Keller was the first to use analysis in these gaps, so-called necks, as an effective tool for estimation of effective properties of particle reinforced composites [14]. For homogenization of other high contrast heterogeneous media Kozlov [15] also advocated for understanding local geometric properties via singular asymptotics. Similarly, we view an understanding of our two-inclusion model as the key first step in the multiscale analysis of more general cases of composites with many particles of different shapes.

Our paper is organized as follows. Section 2 provides problem formulation and statements of the main results. Section 3 presents proofs of the auxiliary Proposition 2.1. Discussion of the linear case ($p = 2$) and comparison with the previous results are given in section 4. The proof of the main theorem is provided in section 5. In the rest of the introduction we review some earlier results on gradient estimates for high contrast composites.

Bonnetier and Vogelius [7] studied elliptic regularity for 2-dimensional problems with discontinuous coefficients that model two-phase fiber-reinforced composites with touching fibers. Material properties of its constituents were assumed to be finite and strictly positive. They showed that solutions are $W^{1,\infty}$ for sufficiently smooth boundary data and conjecture the $\delta^{-1/2}$ blow-up rate (2) for high contrast materials. Li and Vogelius [18] showed $C^{1,\alpha}$-regularity, $0 < \alpha \leq 1/4$, of solutions to elliptic equations that model inhomogeneous materials in $\mathbb{R}^d$. They obtained uniform
bounds on $|\nabla u|$ independent of the distance between inhomogeneities. These bounds depended on their sizes, shapes, and material properties. They further conjectured that this gradient exhibits a singular behavior when inhomogeneities are infinitely conducting.

Li and Nirenberg [17] extended results in [18] to the vectorial case, namely, to linear systems of elasticity. They obtained $C^{1,\alpha}$ interior estimates on domains in $\mathbb{R}^d$ with noncircular inclusions.

Ammari, Kang, and Lim [4] were the first to investigate the case of the close-to-touching regime of particles whose conductivities degenerate, that is, the case of a high contrast composite with perfectly conducting or insulating particles in the background medium of finite conductivity. A lower bound on $|\nabla u|$ was constructed there showing blow-up in both the perfectly conducting and insulating cases. This blow-up was proved to be of order $\delta^{-1/2}$ in $\mathbb{R}^2$, where $\delta > 0$ is the distance between two circular particles. In their subsequent work with H. Lee and J. Lee [2] they established upper and lower bounds on the electric field for the close-to-touching regime of two circular particles in $\mathbb{R}^2$ with degenerate conductivities. This study reveals that the blow-up of the electric field is of order $\delta^{-1/2}$, and in the high contrast regime it occurs at the points of the closest distance between particles. Also considered is the case of a particle close to the boundary for which similar lower and upper bounds for $|\nabla u|$ are established. Essentially 2-dimensional potential theory techniques are used in [2], and the authors point out the importance of the 3-dimensional case.

In [3] Ammari et al. extend results of [2, 4] and decompose the solution into two parts whose gradients are bounded and singular, respectively. This decomposition allows for capturing the gradient's blow-up of the electric field between circular particles in $\mathbb{R}^2$. Also considered in [3] is a case of nonzero permittivity of inclusions whose presence is shown to reduce the blow-up of the gradient.

Yun [27, 28] generalized the blow-up results of [2, 4] to the case of particles of arbitrary sufficiently smooth shape in $\mathbb{R}^2$. Using probabilistic methods it is shown there that the blow-up rate of the electric field in composites with arbitrarily shaped particles is the same as that of disks, that is, of $O(\delta^{-1/2})$. Results [27, 28] were extended by Lim and Yun [20] to the case of spherical particles in $\mathbb{R}^d$, $d \geq 2$. This is the first result where constants in constructed bounds explicitly contain information about geometry of particles.

Recently, in a preprint [13] Kang, Lim, and Yun obtained (3) in the linear case in two dimensions. They further characterize asymptotically the singular part of the solution for both insulating and perfectly conducting circular particles using potential theory.

Bao, Li, and Yin [5] analyzed a model of a composite with two perfectly conducting particles in a bounded domain and away from the external boundary. The optimal upper and lower bounds for the electric field in a composite when the distance between particles is small were obtained. The results of [5], obtained independently of [20, 27, 28], hold for arbitrary shapes of the particles, which are strictly convex in the neighborhood of the point of the shortest distance between particles, and any dimension $d \geq 2$. Bao, Li, and Yin further generalized their results to the case of $N \geq 2$ perfectly conducting particles and to the case of insulating (zero conductivity) particles in [6].

2. Formulation and main results. Let $\Omega \in \mathbb{R}^d$, $d = 2, 3$, be an open bounded domain with $C^{1,\alpha}, 0 < \alpha \leq 1$, boundary $\partial \Omega$. It contains two spherical particles $B^1_\delta$ and $B^2_\delta$ centered at $x_i$, $i = 1, 2$, and at the distance $\delta$ from each other (see Figure 1).
We assume
\[(4) \quad \text{dist}(\partial \Omega, B_1^\delta \cup B_2^\delta) \geq K\]
for some $K$ independent of $\delta$. Let $\Omega_\delta$ model the background medium of the composite, that is, $\Omega_\delta = \Omega^{(B_1^\delta \cup B_2^\delta)}$.

Consider
\[(5) \quad \nabla \cdot (|\nabla u_\delta|^{p-2} \nabla u_\delta) = 0 \quad \text{in} \quad \Omega_\delta,
\]
\[u_\delta = T_\delta^i \quad \text{in} \quad B_i^\delta, \quad i = 1, 2,
\]
\[\int_{\partial B_i^\delta} |\nabla u_\delta|^{p-2} \mathbf{n} \cdot \nabla u_\delta \, ds = 0, \quad i = 1, 2,
\]
\[u_\delta = U(x) \quad \text{on} \quad \partial \Omega,
\]

where a bounded weak solution $u_\delta$ represents the electric potential in $\Omega_\delta$, $p \in \mathbb{N}$, and $p \geq 2$, and $U(x)$ is the given applied potential on the external boundary $\partial \Omega$. Note that $u_\delta$ is constant $T_\delta^i$ on the particle $B_i^\delta$, $i = 1, 2$, which should be found while solving (5).

In order to formulate our main result for (5), we first describe the meaning of the limit in (3). Given two particles $B_i$ in a domain $\Omega$, consider a family of auxiliary problems where the two particles move along the line connecting their centers until they touch. When $B_i$ are located at the distance $\delta > 0$ from each other, we denote them $B_i^\delta$, and this gives us Figure 1 and (5).

When particles touch at $\delta = 0$ we denote them by $B_i^0$; see Figure 2. We further construct a neck $\Pi_0 = \Pi_0(w)$ of a fixed small width $w > 0$ by cutting out a region that contains the touching point of the two particles; see Figure 3(a). We denote by $\varsigma_i$, $i = 1, 2$, the part of $\partial \Pi_0$ that lies on the boundary of the corresponding particle. Similarly, one defines a neck $\Pi_\delta = \Pi_\delta(w)$ and $\varsigma_i$, $i = 1, 2$, when particles are $\delta$-distance apart from each other; see Figure 3(b).

Well-posedness of the limiting problem follows if we look at the variational formulation of (5):
\[(6) \quad u_\delta = \arg\min_{u \in U_\delta} \int_{\Omega_\delta} |\nabla u|^p \, dx, \quad \text{where}
\]
\[U_\delta = \left\{ u \in W^{1,p}(\Omega_\delta) : \quad u|_{B_i^\delta} = t_i, \quad i = 1, 2, \quad u = U(x) \text{ on } \partial \Omega \right\}.
\]
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Fig. 2. Particles’ locations in the limiting problem (7) whose solution is $u_0$.

Fig. 3. (a) The neck $\Pi_0$; (b) the neck $\Pi_\delta$.

The solution of

$$
\nabla \cdot (|\nabla u_0|^{p-2} \nabla u_0) = 0 \quad \text{in } \Omega_0,
$$

$$
u_0 = T_0 \quad \text{in } B_1^0 \cup B_2^0,
$$

is the minimizer of (6) with $\delta = 0$. Note that in this case $u_0$ is the same constant on both particles.

If $\Omega_\delta$ has a $C^{1,\alpha_0}$ boundary ($0 < \alpha_0 \leq 1$), then [19] there exists a positive constant $\alpha = \alpha(\alpha_0, p, d)$ such that the bounded weak solution of (5) satisfies $u_\delta \in C^{1,\alpha}(\bar{\Omega}_\delta)$. The following proposition characterizes convergence of $u_\delta$ to $u_0$ in $C^{1,\alpha}$.

**Proposition 2.1.** Let $u_\delta$ be the solution of (5) and $u_0$ be the solution of (7). Then there is a constant $\alpha > 0$, $\alpha = \alpha(n, p)$, so that

$$
\lim_{\delta \to 0} ||u_\delta - u_0||_{C^{1,\alpha}(K)} = 0
$$

for any compact $K \subset \subset \Omega_0$. Further, for any $i = 1, 2$ and for any neck width $w$

$$
\int_{\partial B_i \setminus \varsigma_i} |\nabla u_\delta|^{p-2} n \cdot \nabla u_\delta = \lim_{\delta \to 0} \int_{\partial B_i \setminus \varsigma_i} |\nabla u_\delta|^{p-2} n \cdot \nabla u_\delta.
$$

While proving Proposition 2.1 we show that for any neck width $w$ there exists $C = C(w, U)$ so that

$$
|\nabla u_\delta(x)| \leq C \quad \text{for any } x \in \bar{\Omega}_\delta \setminus \bar{\Pi}_\delta.
$$

This means that the only possible place for singular behavior of $|\nabla u_\delta|$ is between the particles.

We emphasize that $u_\delta$ and $u_0$ differ in their constraints. Namely, for any $\delta > 0$ the integral of the flux of $u_\delta$ along the boundary of each of the particles is zero; see
the third condition in (5). In contrast, for \( u_0 \) we assume only that the total flux of \( u_0 \) along the boundary of both particles is zero; see the third condition in (7). Generically the quantity

\[
R_0 := \int_{\partial B_0^2} |\nabla u_0|^{p-2} \mathbf{n} \cdot \nabla u_0 \, ds
\]

is not zero. Since \( u_0 \in W^{1,\infty} \), we may use (8) to conclude

\[
|R_0 - \lim_{\delta \to 0} \int_{\partial B_0^2} |\nabla u_\delta|^{p-2} \mathbf{n} \cdot \nabla u_\delta \, ds| \leq C
\]

for any neck width \( w \). If \( R_0 \neq 0 \), we assume without loss of generality that \( R_0 > 0 \). Assumption \( R_0 > 0 \) implies that for sufficiently small \( \delta \) we have the inequality \( T_\delta^2 > T_\delta^3 \). It turns out that \( R_0 = R_0[U] \) is the key characteristic parameter of the \( W^{1,\infty} \) blow-up of \( u_\delta \).

**Theorem 2.2.** Suppose \( d \leq p \). If \( R_0 > 0 \) and \( \gamma = p - (d + 1)/2 \), then

\[
\lim_{\delta \to 0} (T_\delta^2 - T_\delta^3)^{p-1}\delta^{-\gamma} = R_0/C_\delta,
\]

where \( C_\delta \) is an explicitly computable constant given in Table 2. Similarly, if \( d = 3 \) and \( p = 2 \), then

\[
\lim_{\delta \to 0} (T_\delta^2 - T_\delta^3)^{p-1} \ln \delta^{-1} = R_0/C_\delta.
\]

**Table 2**

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<th>( p = 2 )</th>
<th>( p = 3 )</th>
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<tbody>
<tr>
<td>( \pi \sqrt{R} )</td>
<td>( \pi \sqrt{R}/2 )</td>
<td>( 3\pi \sqrt{R/8} )</td>
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<tr>
<td>( \pi \prod_{k=1}^{\infty} \frac{1}{k} \sqrt{R} )</td>
<td>( \pi \sqrt{R/2} )</td>
<td>( \pi \sqrt{R/8} )</td>
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For the sake of brevity, we focus our computations on the case \( d \leq p \) only; the log-case can be treated similarly.

Suppose we rotate and shift the domain \( \Omega_\delta \) so that \( B_0^2 \) is above \( \mathcal{B}_1 \), as depicted in Figure 4. In the proof of Theorem 2.2 we use barriers to show that

\[
\frac{T_\delta^2 - T_\delta^3}{\delta + x^2/R} (1 + O(\delta)) - C \leq \nabla u_\delta(x) \leq \frac{T_\delta^2 - T_\delta^3}{\delta + x^2/R} (1 + O(\delta)) + C, \quad x \in \varsigma_2,
\]

where \( C = C(\max(U(x)), \min(U(x)), K) \) and \( K \) is given in (4). Since \( |\nabla u_\delta| \) satisfies the maximum principle [10], we may use (9) and (12) to immediately obtain the following corollary of Theorem 2.2.

**Corollary 2.3.** Suppose \( d \leq p \). If \( u_\delta \) solves (5), then

\[
\|\nabla u_\delta\|_{L^\infty(\Omega_\delta)} = \frac{|T_\delta^2 - T_\delta^3|}{\delta} [1 + O(\delta)] = \left( \frac{R_0}{C_\delta} \right)^{\frac{1}{p-1}} \delta^{\frac{1}{p-1}} \left[ 1 + O(\delta) \right].
\]

Similarly, if \( d = 3 \) and \( p = 2 \), then

\[
\|\nabla u_\delta\|_{L^\infty(\Omega_\delta)} = \left( \frac{R_0}{C_\delta} \right)^{\frac{1}{2}} \left( \ln \frac{\delta^{-1}}{\delta^{1/p}} \right)^{\frac{1}{2}} [1 + O(\delta)].
\]
3. Proof of Proposition 2.1. Let \( u_\delta \) be the solution of (5). We claim that there exists a constant \( C = C(U) > 0 \) independent of \( \delta \) such that

\[
\|\nabla u_\delta\|_{L^\infty(\partial\Omega)} \leq C.
\]

We prove (14) by constructing upper and lower barriers for \( u \) at \( x_0 \in \partial\Omega_0 \). The upper barrier \( \psi_e \) for \( x_0 \in \partial\Omega_0 \) can be chosen as the solution to the following problem:

\[
\begin{align*}
\nabla \cdot (|\nabla \psi_e|^{p-2}\nabla \psi_e) &= 0 \quad \text{in } \Omega_\delta, \\
\psi_e &= \max_{\partial\Omega} U(x) \quad \text{on } \partial B_1^\delta \cup \partial B_2^\delta, \\
\psi_e &= U(x) \quad \text{on } \partial\Omega.
\end{align*}
\]

This implies there exists a constant \( C_{eu} > 0 \) independent of \( \delta \) such that

\[
|\nabla \psi_e(x_0)| \leq C_{eu}.
\]

Similarly, a lower barrier \( \varphi_e \) for \( x_0 \in \partial\Omega_0 \) is chosen to be the solution of

\[
\begin{align*}
\nabla \cdot (|\nabla \varphi_e|^{p-2}\nabla \varphi_e) &= 0 \quad \text{in } \Omega_\delta, \\
\varphi_e &= \min_{\partial\Omega} U(x) \quad \text{on } \partial B_1^\delta \cup \partial B_2^\delta, \\
\varphi_e &= U(x) \quad \text{on } \partial\Omega.
\end{align*}
\]

and there exists a constant \( C_{el} > 0 \) independent of \( \delta \) such that

\[
|\nabla \varphi_e(x_0)| \leq C_{el}.
\]

Since

\[0 \leq |n \cdot \nabla u_\delta(x)| \leq \max \{|n \cdot \nabla \varphi_e(x)|, |n \cdot \nabla \psi_e(x)|\} \quad \text{if } x_0 \in \partial\Omega,\]

estimates (16) and (18) imply (14).

We now show that for any fixed neck width \( w \) there exists a constant \( C = C(w) \) such that

\[
|\nabla u_\delta(x)| \leq C \quad \text{for } x \in \partial\Pi_\delta^+.
\]
Consider an auxiliary function
\[ \Phi(x) = f(x)|\nabla u_\delta|^2 + \kappa u_\delta^3, \quad x = (x, y) \in \Omega. \]

This proof focuses on two dimensions; the 3-dimensional case follows similarly. Choose a function \( f(x) \) defined on the coordinate system as in Figure 4 to be as follows:

\[
 f(x) = \begin{cases} 
 0, & |x| \leq \frac{w}{2}, \\
 \frac{8}{w^2} \left( |x| - \frac{w}{2} \right)^2, & \frac{w}{2} < |x| < \frac{3}{4} w, \\
 \frac{8}{w^2} |x|^2 + 1, & \frac{3}{4} w^2 < |x| < w, \\
 1, & |x| \geq w.
\end{cases}
\]

There exists a constant \( \kappa = O(1/w^2) \) so that \( \Phi \) satisfies the maximum principle [9, 21]. Then

\[
\max_{\partial \Omega^\pm} |\nabla u_\delta|^2 \leq \max_{\Omega} \Phi(x, y) \leq C(w) + \max_{\partial B_1^{0, \delta}} f(x)|\nabla u_\delta|^2.
\]

Since \( |\nabla u_\delta| = O(1) \) on \( \partial \Omega \) by (14), we need only obtain estimates for \( f(x)|\nabla u_\delta|^2 \) on \( \partial B_1^{i, \delta} \), \( i = 1, 2 \), for \( 1/2 < w < |x| \). This can be shown by constructing the upper and lower barriers for \( u_\delta \) at \( x \in \zeta_i \), \( i = 1, 2 \). It is straightforward to verify that

\[
\psi(x) = \begin{cases} 
 a|x - x_0|^\beta + b, & d \neq p, \\
 a \log |x - x_0| + b, & d = p,
\end{cases}
\]
is the solution of the \( p \)-Laplace equation

\[
\nabla \cdot (|\nabla \psi|^{p-2} \nabla \psi) = 0 \quad \text{on} \quad \mathbb{R}^d \setminus x_0
\]
for any \( x_0, a, b, \) and \( \beta = (p - d)/(p - 1) \). Once again, for brevity we focus our computations on the case \( d \neq p \) only; the log-case can be treated similarly.

The upper barrier for \( u_\delta \) at \( x = (x, y) \in \zeta_1 \) is obtained as follows. Construct a circle \( C_1 \) of the radius \( r_1 = w/100 \) (see Figure 7) centered at a point inside \( B_1^1 \) that touches \( \partial B_1^1 \) at \( x \in \zeta_1 \). Construct a concentric to \( C_1 \) circle denoted by \( C_2 \) of a radius \( r_2 = r_2(r_1) \) that touches \( \partial B_2^1 \), and

\[
r_2 = \delta + r_1 + \frac{1}{2R} \left( \frac{R - r_1}{R} \right) \left( \frac{2R - r_1}{R} \right) x^2.
\]

Then an upper barrier \( \overline{\sigma} \) for \( u_\delta \) at \( x \in \zeta_1 \) is obtained using the radial solution (22) that satisfies \( \overline{\sigma}(r_1) = T_1^1, \overline{\sigma}(r_2) = \max_{\partial \Omega} U(x) \), and the lower barrier \( \underline{\sigma} \) for \( u_\delta \) at \( x \in \zeta_1 \) is obtained using the radial solution (22) that satisfies \( \underline{\sigma}(r_1) = T_1^1, \underline{\sigma}(r_2) = \min_{\partial \Omega} U(x) \). Then

\[
|\nabla u_\delta(x)| = |(\nabla u_\delta \cdot n)| \leq \max \{ C, |\nabla \overline{\sigma}, |\nabla \underline{\sigma}| \}, \quad x \in \zeta_1,
\]
where

\[
|\nabla \overline{\sigma}| = \left| \frac{d\overline{\sigma}}{dr} \right| = \max_{\partial \Omega} U(x) - T_2^1 \left( \frac{\beta r_1^{\beta-1}}{r_2^{\beta} - r_1^{\beta}} \right), \quad |\nabla \underline{\sigma}| = \left| \frac{d\underline{\sigma}}{dr} \right| = |T_2^1 - \min_{\partial \Omega} U(x)| \frac{\beta r_1^{\beta-1}}{r_2^{\beta} - r_1^{\beta}}.
\]
Since \( r_2 = r_1 + O(w) \), then \( r_2^\beta - r_1^\beta = O(w) \); hence, we have
\[
|\nabla u_\delta(x)| \leq C, \quad C = C(w), \quad x = (x, y) \in \varsigma_1, \quad \text{and} \quad \frac{w}{2} < |x| < R.
\]

Similarly, one can bound the gradient at \( x \in \varsigma_2 \). Substituting these estimates back into (21), we have that there exists \( C = C(w) \) such that (19) holds.

The maximum principle and (21) imply (9), which states
\[
|\nabla u_\delta(x)| \leq C, \quad C = C(w), \quad x = (x, y) \in \varsigma_1, \quad \text{and} \quad w^2 < |x| < R.
\]

Similarly, one can bound the gradient at \( x \in \varsigma_2 \). Substituting these estimates back into (21), we have that there exists \( C = C(w) \) such that (19) holds.

The maximum principle and (21) imply (9), which states
\[
|\nabla u_\delta(x)| \leq C \quad \text{for any} \quad x \in \Omega_\delta \setminus \Pi_\delta.
\]

Now the estimate (25) means that, up to a subsequence, \( u_\delta \to u_* \) as \( \delta \to 0 \) strongly in \( W^{1,q} \) for any \( q < \infty \). We claim that \( u_0 = u_* \), where \( u_0 \) solves (7). Indeed, the only condition that needs to be verified is
\[
\int_{\partial B_1^0} |\nabla u_*|^{p-2} n \cdot \nabla u_* \, ds + \int_{\partial B_2^0} |\nabla u_*|^{p-2} n \cdot \nabla u_* \, ds = 0.
\]

For any small parameter \( \varepsilon > 0 \) construct a domain \( K_\varepsilon \) with a \( C^\infty \) boundary such that it approximates \( B_1^0 \cup B_2^0 \) arbitrarily well; that is, each point on \( \partial K_\varepsilon \) is located at a distance \( O(\varepsilon) \) from \( \partial (B_1^0 \cup B_2^0) \); see Figure 5.

From [26] we know that there is a constant \( \alpha > 0 \), \( \alpha = \alpha(n, p, \alpha_0) \), such that
\[
||u_\delta||_{C^{1,\alpha}} \leq C
\]
on \( \overline{\Omega_0} / K_\varepsilon \) uniformly in sufficiently small \( \delta \). We therefore obtain that \( u_\delta \to u_* \) strongly in \( C^1(\overline{\Omega_0} / K_\varepsilon) \). Integrating the first equation in (5) over \( K_\varepsilon / (B_1^0 \cup B_2^0) \) and using the third equation in (5), we obtain
\[
\int_{\partial K_\varepsilon} |\nabla u_\delta|^{p-2} n \cdot \nabla u_\delta \, ds = 0.
\]

Since \( u_\delta \to u_* \) strongly in \( C^1(\overline{\Omega_0} / K_\varepsilon) \), we have
\[
\int_{\partial K_\varepsilon} |\nabla u_*|^{p-2} n \cdot \nabla u_* \, ds = 0.
\]

Since \( \varepsilon \) is arbitrary, we obtain (26). Using [26] \( u_0 \) is the unique \( C^{1,\alpha}(\Omega_0) \) solution of (7). Uniqueness of \( u_0 \) allows us to conclude that there is a pointwise convergence \( u_\delta \to u_* = u_0 \) for all \( \delta \to 0 \). From [26] we also conclude that the convergence is in \( C^{1,\alpha}(K) \), \( K \subset \subset \Omega \) as \( \delta \to 0 \).
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Fig. 6. (a) The smooth curve $\Gamma = \mu \cup \eta$ and its flat piece $\mu$; (b) the portion $\varsigma_2$ of $\partial B^2_\delta$ that belongs to the neck $\Pi_\delta$; (c) the half-neck $\frac{1}{2} \Pi_\delta$.

Proof of (8) follows, once we recall that outside any neck $u_\delta$ and $u_0$ are uniformly bounded in $W^{1,\infty}$. So we can take the family of contours depicted in Figure 6(a)–6(b), integrate the $p$-Laplacian in the interior of these contours, and conclude that the integrals over the boundary of the particles are equal (up to a small constant) to integrals in the interior, where we have convergence. Let us provide more detail.

We first construct a closed $C^\infty$-curve $\Gamma$ that contains $B^0_2$ and passes through the point where two circles $B^0_2$ and $B^0_1$ touch. This curve is large enough to contain $B^0_2$ inside when particles are “moved away” from one another such that the distance between any two is $\delta$ (recall that in such a configuration particles are denoted by $B^1_\delta$ and $B^2_\delta$). We also assume that this curve has a flat portion of the length $2w$ denoted by $\mu$ which is perpendicular to the line connecting the two particles, and the rest of the curve is $\eta = \Gamma \setminus \mu$; see Figure 6(a). Then

$$R_0 = \int_\mu |\nabla u_0|^{p-2}(\mathbf{n} \cdot \nabla u_0) \, ds + \int_\eta |\nabla u_0|^{p-2}(\mathbf{n} \cdot \nabla u_0) \, ds.$$  

Since $|\nabla u_0| < C$ everywhere in $\Omega_0$ [7], we have

$$|R_0 - \int_\eta |\nabla u_0|^{p-2}(\mathbf{n} \cdot \nabla u_0) \, ds| = \left| \int_\mu |\nabla u_0|^{p-2}(\mathbf{n} \cdot \nabla u_0) \, ds \right| \leq Cw.$$  

We now move particles away from each other such that the distance between them is $\delta$ (see Figure 6(b)) and consider a region between the two curves $\partial B^2_\delta$ and $\Gamma$ where we still have $\nabla \cdot |\nabla u_\delta|^{p-2}\nabla u_\delta = 0$. Multiplying this equation by $u_\delta$ and integrating by parts, we obtain

$$\int_\mu |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_\delta) \, ds + \int_\eta |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_\delta) \, ds - \int_{\partial B^2_\delta} |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_\delta) \, ds = 0,$$  

where the third integral of the left-hand side is zero. This integral we split into the piece that belongs to the neck $\Pi_\delta$ denoted as above by $\varsigma_2$ and the rest of the boundary $\partial B^2_\delta \setminus \varsigma_2$. Then we have

$$\int_{\partial B^2_\delta \setminus \varsigma_2} |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_\delta) \, ds = \int_\mu |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_\delta) \, ds$$  

$$+ \int_\eta |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_\delta) \, ds - \int_{\varsigma_2} |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_\delta) \, ds.$$
Since we have a pointwise convergence of \( u_\delta \) to \( u_0 \) in any compact \( K \subset \subset \Omega \setminus (B^1_\delta \cup B^2_\delta) \), due to [26] we have that

\[
(31) \quad \int_\eta |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_\delta) \, ds \to \int_\eta |\nabla u_0|^{p-2}(\mathbf{n} \cdot \nabla u_0) \, ds \quad \text{as} \quad \delta \to 0.
\]

Now we multiply the equation \( \nabla \cdot |\nabla u_\delta|^{p-2}\nabla u_\delta = 0 \) in the region between \( \mu \) and \( \varsigma^2 \) and integrate by parts. We obtain that

\[
(32) \quad \int_\mu |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_\delta) \, ds - \int_\varsigma^2 |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_\delta) \, ds + \int_{\frac{1}{2}\overline{\Omega}_\delta^+} |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_\delta) \, ds = 0,
\]

where \( \frac{1}{2}\overline{\Omega}_\delta^+ \) (see Figure 6(c)) is the half of the lateral boundary of the neck attached to \( B^2_\delta \). So, now substituting (32) into (30) and using (31), we have

\[
\left| \int_{\partial B^1_\delta \setminus \varsigma^2} |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_\delta) \, ds - \int_{\partial B^2_\delta \setminus \varsigma^2} |\nabla u_0|^{p-2}(\mathbf{n} \cdot \nabla u_0) \, ds \right|
= \left| \int_\eta |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_\delta) \, ds \right|
- \left| \int_{\frac{1}{2}\overline{\Omega}_\delta^+} |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_\delta) \, ds \right|
- \left| \int_{\partial B^2_\delta \setminus \varsigma^2} |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_0) \, ds \right| + o(\delta)
\leq \left| \int_{\frac{1}{2}\overline{\Omega}_\delta^+} |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_\delta) \, ds \right|
- \left| \int_{\frac{1}{2}\overline{\Omega}_\delta^+} |\nabla u_0|^{p-2}(\mathbf{n} \cdot \nabla u_0) \, ds \right| + o(\delta) \to 0
\]
as \( \delta \to 0 \). This estimate leads to (8). Finally, substituting (31)–(32) into (30) and using (28), we obtain

\[
\lim_{\delta \to 0} \left| R_0 - \int_{\partial B^2_\delta \setminus \varsigma^2} |\nabla u_\delta|^{p-2}(\mathbf{n} \cdot \nabla u_\delta) \, ds \right|
= \left| \int_{\frac{1}{2}\overline{\Omega}_\delta^+} |\nabla u_0|^{p-2}(\mathbf{n} \cdot \nabla u_0) \, ds \right| \leq Cw. \quad \Box
\]

4. Linear case \( p = 2 \). For \( p = 2 \) we chose to compare our results with [5], because [5] is easy to interpret in terms of our problem. As mentioned in [5], the results [2, 4, 20, 27, 28] are essentially the same. Consider a functional

\[
Q_\delta[U] = \int_{\partial B^1_\delta} \frac{\partial v_3}{\partial n} \int_{\partial \Omega} \frac{\partial v_1}{\partial n} - \int_{\partial B^2_\delta} \frac{\partial v_3}{\partial n} \int_{\partial \Omega} \frac{\partial v_1}{\partial n} \quad \text{and} \quad a_{ij} = \int_{\partial B^1_\delta} \frac{\partial v_j}{\partial n},
\]

where \( v_i \) (\( i = 1, 2, 3 \)) solve the following problems:

\[
\begin{align*}
\Delta v_1 &= 0 \quad \text{in} \quad \Omega_\delta, & \Delta v_2 &= 0 \quad \text{in} \quad \Omega_\delta, & \Delta v_3 &= 0 \quad \text{in} \quad \Omega_\delta, \\
v_1 &= 1 \quad \text{on} \quad \partial B^1_\delta, & v_2 &= 1 \quad \text{on} \quad \partial B^2_\delta, & v_3 &= 0 \quad \text{on} \quad \partial B^1_\delta \cup \partial B^2_\delta, \\
v_1 &= 0 \quad \text{on} \quad \partial B^1_\delta \cup \partial \Omega, & v_2 &= 0 \quad \text{on} \quad \partial B^2_\delta \cup \partial \Omega, & v_3 &= U \quad \text{on} \partial \Omega.
\end{align*}
\]
The main result, Theorems 1.1 and 1.2 of [5], states that
\[ C|Q_\delta[U]| \leq |T_\delta^2 - T_\delta^3| \leq C(U) \sqrt{\delta}, \quad C|Q_\delta[U]| \leq |T_\delta^2 - T_\delta^3| \leq \frac{C(U)}{\ln \delta}, \]
for \( d = 2, 3 \), respectively. Thus \( Q_\delta[U] \) is a characteristic parameter that determines blow-up. Let us relate it to our \( R_\delta \). We verify that
\[ Q_\delta[U] = \int_{\partial B_1^\delta} \frac{\partial v_3}{\partial n} \int_{\partial \Omega} \frac{\partial v_2}{\partial n} - \int_{\partial B_2^\delta} \frac{\partial v_3}{\partial n} \int_{\partial \Omega} \frac{\partial v_1}{\partial n}, \]
where \( v_i \) (\( i = 1, 2 \)) are as above and \( v_3 \) solves
\[ \nabla \cdot |\nabla v_3|^{p-2} \nabla v_3 = 0 \quad \text{in } \Omega_\delta, \]
\[ v_3 = T_\delta \quad \text{in } B_1^\delta \cup B_2^\delta, \]
\[ \int_{\partial B_1^\delta} |\nabla v_3|^{p-2} \nabla v_3 \cdot n \, ds + \int_{\partial B_2^\delta} |\nabla v_3|^{p-2} \nabla v_3 \cdot n \, ds = 0, \]
\[ v_3 = U(x) \quad \text{on } \partial \Omega \]
with \( p = 2 \). Indeed, we have that
\[ v_3 = T_\delta [v_1 + v_2] + v_3 \quad \text{in } \Omega_\delta; \]
then
\[ \int_{\partial B_1^\delta} \frac{\partial v_3}{\partial n} \int_{\partial \Omega} \frac{\partial v_2}{\partial n} - \int_{\partial B_2^\delta} \frac{\partial v_3}{\partial n} \int_{\partial \Omega} \frac{\partial v_1}{\partial n} \]
\[ = T_\delta \left[ \int_{\partial B_1^\delta} \frac{\partial v_1}{\partial n} + \int_{\partial B_1^\delta} \frac{\partial v_2}{\partial n} \right] \int_{\partial \Omega} \frac{\partial v_2}{\partial n} - T_\delta \left[ \int_{\partial B_2^\delta} \frac{\partial v_1}{\partial n} + \int_{\partial B_2^\delta} \frac{\partial v_2}{\partial n} \right] \int_{\partial \Omega} \frac{\partial v_1}{\partial n} + Q_\delta[U] \]
\[ = T_\delta (a_{11} + a_{12})(-a_{12} - a_{22}) - T_\delta (a_{21} + a_{22})(-a_{11} - a_{21}) + Q_\delta[U] = Q_\delta[U], \]
using \( a_{12} = a_{21} \). Since
\[ \int_{\partial B_1^\delta} \frac{\partial v_3}{\partial n} = -\int_{\partial B_2^\delta} \frac{\partial v_3}{\partial n}, \]
we conclude that
\[ Q_\delta[U] = -\left( \int_{\partial \Omega} \frac{\partial v_2}{\partial n} + \int_{\partial \Omega} \frac{\partial v_1}{\partial n} \right) R_\delta[U], \]
where
\[ R_\delta[U] := \int_{\partial B_2^\delta} |\nabla v_3|^{p-2} \nabla v_3 \cdot n \, ds, \quad p = 2. \]
By the maximum principle
\[ 0 < C_1 \leq -\left( \int_{\partial \Omega} \frac{\partial v_2}{\partial n} + \int_{\partial \Omega} \frac{\partial v_1}{\partial n} \right) \leq C_2; \]
therefore, the asymptotic behavior of $Q_\delta[U]$ is the same as the asymptotic behavior of $R_\delta[U]$. Using methods such as those in section 2.1, we can show the following.

**Proposition 4.1.** Let $v_\delta$ be the solution of (33) and $u_0$ be the solution of (7). Then there is a constant $\alpha > 0$, $\alpha = \alpha(d, p)$, so that
\[
\lim_{\delta \to 0} ||v_\delta - u_0||_{C^{1, \alpha}(K)} = 0
\]
for any compact $K \subset \subset \Omega_0$. Further,
\[
(35) \quad \lim_{\delta \to 0} R_\delta = R_0.
\]

5. **Proof of Theorem 2.2.** We will use here the method of barriers and the radial solution (22) to prove an estimate, which is slightly stronger than (12).

**Lemma 5.1.** For any constants $T_1^\delta$ and $T_2^\delta$, $m \leq T_1^\delta \leq T_2^\delta \leq M$, there exists $\delta_0$ such that for any $\delta \leq \delta_0$ the solution of
\[
\nabla \cdot (|\nabla u_\delta|^{p-2}\nabla u_\delta) = 0 \quad \text{in } \Omega_\delta,
\]
\[
(36) \quad u_\delta = T_1^\delta \quad \text{in } B_\delta^1, \ i = 1, 2,
\]
\[
\text{and } u_\delta = U(x) \quad \text{on } \partial\Omega
\]
satisfies an estimate
\[
(37) \quad T_2^\delta - T_1^\delta \frac{1 + O(\delta)}{\delta + x^2/R} (1 + \delta) - C \leq \nabla u_\delta(x) \leq T_2^\delta - T_1^\delta \frac{1 + O(\delta)}{\delta + x^2/R} (1 + \delta) + C, \quad x \in \zeta_2,
\]
where $C = C(\max(U(x)), \min(U(x)), K)$ and $K$ is given in (4).

In contrast to (5), the constants $T_1^\delta$ and $T_2^\delta$ in (36) are arbitrary. This implies that the solution of (36) may not satisfy the integral identities for the flux of $u_\delta$ on $\partial B_\delta^1$ as in (5).

**Proof.** An upper barrier $\overline{v}$ for $u_\delta$ at the point $(x, y) \in \zeta_1$ is constructed as follows. Consider a circle of radius $0 < r_1 < R$ that touches the circle $\partial B_\delta^1$ from within (as in Figure 7) at the point $(x, y)$. Also, another circle of radius $r_2 > r_1$ is considered that is centered at the same point as the other one and touches the boundary $\partial B_\delta^1$. Then
\[
(38) \quad r_2 = r_2(x, r_1) = \delta + r_1 + \frac{1}{2} \left(1 - \frac{r_1}{R}\right) \left(2 - \frac{r_1}{R}\right) \frac{x^2}{R}.
\]
Choosing $r_1 = \delta$, we obtain
\[
(39) \quad r_2 - r_1 = \left(\delta + \frac{x^2}{R}\right) (1 + O(\delta)).
\]

Then we construct an upper barrier $\overline{v}$ for $u_\delta$ at the point $(x, y) \in \zeta_1$ via radial solution (22) that takes values
\[
\overline{v}(r_1) = T_1^\delta, \quad \overline{v}(r_2) = T_2^\delta.
\]

Then
\[
(40) \quad \nabla u_\delta(x, y) \leq \frac{d\overline{v}}{dr}(r_1) = \beta_1^{\delta - 1} \frac{T_2^\delta - T_1^\delta}{r_2 - r_1}.
\]
By the mean-value property

\[ r_2^\beta - r_1^\beta = \beta r_0^{\beta - 1}(r_2 - r_1), \quad r_1 < r_0 < r_2, \]

and therefore

\[ n \cdot \nabla u_\delta(x,y) \leq \beta r_0^{\beta - 1}\frac{T_\delta^2 - T_\delta^1}{r_2 - r_1} = \left( \frac{r_1}{r_0} \right)^{\beta - 1}\frac{T_\delta^2 - T_\delta^1}{r_2 - r_1} \]
\[ \leq \frac{T_\delta^2 - T_\delta^1}{r_2 - r_1} = \frac{T_\delta^2 - T_\delta^1}{\delta + x^2/R} (1 + O(\delta)). \]

We can find \( \delta_0 \) and \( C = C(K,m,M,w,\delta_0) \) so that if \( (T_\delta^2 - T_\delta^1)/(r_2 - r_1) \geq s \) and \( \delta \leq \delta_0 \), then

\( u_\delta(x) \leq \mathfrak{U}(x) \) for all \( x \in \partial \Omega \).

Thus \( \mathfrak{U} \) is an upper barrier if \( (T_\delta^2 - T_\delta^1)/(r_2 - r_1) \geq s \) and \( \delta \leq \delta_0 \). The upper bound in (37) now follows since for other values of \( (T_\delta^2 - T_\delta^1)/(r_2 - r_1) \) and \( \delta \) we estimate

\( n \cdot \nabla u_\delta(x,y) \leq C. \)

For the lower barrier \( \mathfrak{V} \) at \( (x,y) \in \varsigma_1 \) we consider a circle of a small radius \( \rho_1 \) that touches the circle \( \partial B_\delta^1 \) from within and whose center is located on the line connecting the point \( (x,y) \) and the center of \( B_\delta^1 \) (as in Figure 8). Denote the distance from the center \( (\xi, -\eta) \) of this constructed circle of radius \( \rho_1 \) to the point \( (x,y) \) by \( \rho_2 \). Then

\[ \xi = \frac{R + \rho_2}{R} x, \quad (R + \rho_2)^2 = \xi^2 + \left( R + \frac{\delta}{2} + \eta \right)^2, \quad (R - \rho_1)^2 = \xi^2 + \left( R + \frac{\delta}{2} - \eta \right)^2, \]

and we have

\[ \rho_2 = \rho_2(x, \rho_1) = -R + (2R + \delta) \left[ \sqrt{1 - \frac{x^2}{R^2}} - \sqrt{\left( R - \rho_1 \right)^2 - \frac{x^2}{R^2}} \right] \]
\[ = \delta + \rho_1 + \frac{2R + \delta}{2R} \left[ \frac{2R + \delta}{R - \rho_1} - 1 \right] \frac{x^2}{R} + O\left( \frac{x^4}{R^2} \right). \]
Fig. 8. Lower barrier construction at the point \((x, y) \in s_1\).

Choosing \(\rho_1 = \delta\), we obtain

\[
\rho_2 - \rho_1 \leq \begin{cases} 
\left(\delta + \frac{x^2}{R}\right)(1 + O(\delta)), & |x| \leq \delta^{1/4}, \\
C, & |x| \geq \delta^{1/4}.
\end{cases}
\]

Then we construct a lower barrier \(v\) for \(u_\delta\) at the point \((x, y) \in s_1\) via radial solution (22) that takes values 

\[v(\rho_1) = T_\delta^2, \quad v(\rho_2) = T_\delta^1.\]

Then

\[\mathbf{n} \cdot \nabla u_\delta(x, y) \geq -\frac{dv}{dr}(\rho_2) \geq \beta \rho_2^{-1} \frac{T_\delta^2 - T_\delta^1}{\rho_2 - \rho_1} \geq \frac{T_\delta^2 - T_\delta^1}{\rho_2 - \rho_1} \geq \frac{T_\delta^2 - T_\delta^1}{\rho_2 - \rho_1}.\]

Using (40) in (41), we obtain

\[\mathbf{n} \cdot \nabla u_\delta(x, y) \geq \max \left(\frac{T_\delta^2 - T_\delta^1}{\delta + \frac{x^2}{R}} (1 + O(\delta)), C\right).\]

As with the upper bound we can find \(\delta_0\) and \(C = C(K, m, M, w, \delta_0)\) so that if \((T_\delta^2 - T_\delta^1)/(\rho_2 - \rho_1) \geq s\) and \(\delta \leq \delta_0\), then

\[u_\delta(x) \geq v(x) \text{ for all } x \in \partial \Omega.\]

Thus \(v\) is a lower barrier if \((T_\delta^2 - T_\delta^1)/(\rho_2 - \rho_1) \geq s\) and \(\delta \leq \delta_0\). The lower bound in (37) now follows since for other values of \((T_\delta^2 - T_\delta^1)/(\rho_2 - \rho_1)\) and \(\delta\) we again estimate

\[\mathbf{n} \cdot \nabla u_\delta(x, y) \geq -C. \quad \Box\]

We now prove Theorem 2.2.

Proof. From (37) we estimate

\[
(1 + O(\delta)) \int_{-w}^{w} \left(\frac{T_\delta^2 - T_\delta^1}{\delta + \frac{x^2}{R}} \right)^{p-1} dx - Cw \leq \int_{s_1} |\nabla u_\delta|^{p-2} \mathbf{n} \cdot \nabla u_\delta \, ds \\
\leq (1 + O(\delta)) \int_{-w}^{w} \left(\frac{T_\delta^2 - T_\delta^1}{\delta + \frac{x^2}{R}} \right)^{p-1} dx + Cw.
\]
Since for any \( w > 0 \) [21]
\[
\lim_{\delta \to 0} \int_{-w}^{w} \frac{dx}{(\delta + x^2/R)^{p-1}} = C_0,
\]
the claim of Theorem 2.2 follows.

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