Solutions, hw 12

1. It is sufficient to show that the family is not equicontinuous at one point, say $x = 0$. If it were equicontinuous, then for any $\varepsilon > 0$, say $\varepsilon = \frac{1}{2}$ there was $\delta > 0$, such that for all $x$, $0 < x < \delta$ we have

$$|\sin n0 - \sin nx| = |\sin nx| < \frac{1}{2}.$$ 

But, clearly, for any $\delta > 0$ there is an $N$ such that $x = \pi/(2N) < \delta$. But then for the found $N$ and $x$ we have

$$|\sin Nx| = |\sin \pi/2| = 1 > \frac{1}{2}.$$ 

Contradiction. Hence the family is not equicontinuous.

2. a) We have that

$$|F(x)| \leq \int_0^x |f(t)| dt \leq \int_0^x dt \leq \int_0^1 dt = 1.$$ 

Hence for any $F \in \mathcal{F}$ and any $x \in [0, 1]$

$$|F(x)| \leq 1,$$ 

so $\mathcal{F}$ is bounded. Since

$$|F(x) - F(y)| \leq | \int_y^x f(t) dt | \leq | \int_y^x dt | \leq | x - y |.$$ 

We have that all $F \in \mathcal{F}$ are Lipschitz with Lipschitz constant at most 1. We proved in class, that the set of Lipschitz functions with a bounded Lipschitz constant is equicontinuous, and therefore $\mathcal{F}$ is equicontinuous.

Note one useful fact that immediately follows from the last inequality. Let

$$\mathcal{L} = \{ F : [0, 1] \to \mathbb{R}, \ F(0) = 0, \ |F(x) - F(y)| \leq | x - y | \}$$ 

We have already proved that $\mathcal{F} \subset \mathcal{L}$. We know about $\mathcal{L}$ (from lectures), that $\mathcal{L}$ is closed, in fact it is even compact. Therefore the closure of $\mathcal{F}$ must be a subset of $\mathcal{L}$:

$$\overline{\mathcal{F}} \subset \mathcal{L}.$$ 

b) Let

$$f_n = \begin{cases} 
1, 0 \leq x \leq 1/2 - 1/n, \\
-n(x - 1/2), 1/2 - 1/n \leq x \leq 1/2 + 1/n, \\
-1, 1/2 + 1/n \leq x \leq 1.
\end{cases}$$ 

Then

$$F_n \to F = \begin{cases} 
x, 0 \leq x \leq 1/2, \\
1 - x, 1/2 \leq x \leq 1.
\end{cases}$$ 

$F \notin \mathcal{F}$, because it is not continuously differentiable. Hence $\mathcal{F}$ is not closed.

c) By the last remark in part a) we only need to show that

$$\mathcal{L} \subset \overline{\mathcal{F}}.$$
That is we need to show that for any \( f \in \mathcal{L} \) and any \( \varepsilon > 0 \), there exists \( g \in \mathcal{F} \), such that
\[
d(f, g) < \varepsilon.
\]
Let us make a few further simplifications. Let
\[
\mathcal{P} = \{ F : [0, 1] \to \mathbb{R}, \; F(0) = 0, \; |F(x) - F(y)| \leq |x - y|, \; F(x) \text{ is piece-wise linear} \}
\]
Observe that, similar to piece-wise linear approximation Theorem 3.21, for any \( f \in \mathcal{L} \) and any \( \varepsilon > 0 \), there exists \( h \in \mathcal{P} \), such that
\[
d(f, h) < \varepsilon/2.
\]
Hence we really only need to show that a piece-wise linear Lipchitz (with Lipschitz constant at most 1) function can be approximated by a function from \( \mathcal{F} \): for any \( h \in \mathcal{P} \) and any \( \varepsilon > 0 \), there exists \( g \in \mathcal{F} \), such that
\[
d(h, g) < \varepsilon/2.
\]
Let us further simplify the problem Suppose \( h \in \mathcal{P} \). Then \( h' \) is defined everywhere on \([0, 1]\), but at finitely many points \( 0 = x_0 < x_1 < x_2 < \cdots < x_k = 1 \), and we have that the derivative is a piecewise constant function:
\[
h'(x) = a_i, i = 1, 2, \ldots, k, x_{i-1} < x < x_i
\]
We can write
\[
h'(x) = \sum_{i=1}^{k} a_i \chi_{(x_{i-1}, x_i)}, \; a_i \neq 0,
\]
where (recall homework 9) we used the notation
\[
\chi_{(x_{i-1}, x_i)}
\]
is the characteristic function of interval \((x_{i-1}, x_i)\). Our final simplification is analogous to the simplification in the proof of the Weierstrass approximation theorem (see Lemma 7.3). Namely, since \( h' \) is a finite linear combination of step functions, it is sufficient to prove that if
\[
h(x) = \int_{0}^{x} \chi_{(x_{i-1}, x_i)}dx,
\]
then for any
\[
\varepsilon := \frac{\varepsilon}{2|a_i|k} > 0,
\]
there exists \( g \in \mathcal{F} \), such that
\[
d(h, g) < \varepsilon.
\]
The proof of the last claim is as follows. Let approximate the derivative of the step function by a sequence of piecewise linear functions:
\[
g_n(x) = \int_{0}^{x} \tau_n(s)ds, \; \tau_n(x) = \begin{cases} 0, x \in [0, x_{i-1} - 1/n] \cup [x_i + 1/n, 1], \\ 1, x \in [x_{i-1} + 1/n, x_i - 1/n], \\ 1/2 + n/2(x - x_{i-1}), x \in [x_{i-1} - 1/n, x_{i-1} + 1/n], \\ 1/2 - n/2(x - x_i), x \in [x_i - 1/n, x_i + 1/n]. \end{cases}
\]
Then
\[|g_n(x) - h(x)| \leq \int_0^1 |\chi_{(x_{i-1}, x_i)} - \tau_n| dx \leq \int_{x_{i-1}-1/n}^{x_{i-1}+1/n} \frac{1}{2} dx + \int_{x_{i-1}+1/n}^{x_{i}+1/n} \frac{1}{2} dx = \frac{2}{n}.\]

Clearly all \(g_n \in \mathcal{F}\). Therefore for any \(\tilde{\varepsilon} > 0\) there exists \(n\) such that
\[d(g_n, h(x)) = \sup_{x \in [0, 1]} |g_n(x) - h(x)| < \tilde{\varepsilon}.\]

Set \(g = g_n\), and we have proved the claim.

3. Observe that we really need to prove only the following fact: denote \(G : C([0, 1], \mathbb{R}) \to \mathbb{R},\)
\[G(f) = \int_0^1 f(x) dx.\]

Then \(G\) is continuous. Indeed if we knew that \(G\) is continuous, then the proof is immediate: \(G\) is a continuous function from a compact set \(\mathcal{F}\) to real line. Therefore (by Corollary 6.57) \(G\) takes a maximum value on \(\mathcal{F}\) that is there exists \(g \in \mathcal{F}\) such that
\[G(g) \geq G(f).\]

But the last inequality means exactly what we want:
\[\int_0^1 g(x) dx \geq \int_0^1 f(x) dx\]
for all \(f \in \mathcal{F}\).

So let us prove the key observation, that \(G\) is continuous. Set \(\delta = \varepsilon\). Then for every \(\varepsilon > 0\), if
\[d(f, g) < \varepsilon,\]
then
\[|G(g) - G(f)| = |\int_0^1 (g - f) dx| \leq \int_0^1 |g - f| dx < \int_0^1 \varepsilon dx = \varepsilon.\]
Hence \(G\) is continuous.

4. Since \(\mathcal{F}\) is equicontinuous, for every \(x \in K\) there exists \(U_x\) - an open set, \(x \in U_x\), such that for every \(y \in U_x\) and every \(f \in \mathcal{F}\)
\[|f(x) - f(y)| < 1.\]

The above inequality implies that for every \(y \in U_x\) and every \(f \in \mathcal{F}\)
\[m_x - 1 < f(y) < M_x + 1,\]
where both numbers \(M_x\) and \(m_x\) are finite by assumption of the problem. Clearly,
\[\mathcal{U} = \{U_x, x \in K\}\]
is an open cover of \(K\). Since \(K\) is compact there exists a finite subcover
\[K \subset \bigcup_{i=1}^n U_{x_i}.\]

Hence for any \(y \in K\) there exists \(x_i\) such that for any \(f \in \mathcal{F}\)
\[|f(x_i) - f(y)| < 1,\]
which implies
\[ m_{x_i} - 1 < f(y) < M_{x_i} + 1 \]

Finally, since there are only finitely many \( U_{x_i} \) the quantities
\[
M = \max_{i=1,\ldots,n} (M_{x_i}) + 1, \quad m = \min_{i=1,\ldots,n} (M_{x_i}) - 1,
\]
are finite numbers and we have for any \( y \in K \) and for any \( f \in \mathcal{F} \)
\[
m < f(y) < M.
\]

Hence \( \mathcal{F} \) is bounded.