Solutions, hw 11
1. We already know that

\[ d : C(X, Y) \times C(X, Y) \to \mathbb{R}. \]

Let us verify that \( d(f, g) \) satisfies the three properties of a metric.
a) property \( d(f, g) \geq 0, d(f, g) = 0 \) if and only if \( f = g \).
Since for any \( y_1 \in Y, y_2 \in Y \) \( \rho(y_1, y_2) \geq 0 \) we have

\[ d(f, g) = \sup_{x \in X} \rho(f(x), g(x)) \geq 0. \]

Suppose \( d(f, g) = 0 \), that is

\[ \sup_{x \in X} \rho(f(x), g(x)) = 0. \]

It implies that

\[ \rho(f(x), g(x)) \leq 0. \]

Since all \( \rho(f(x), g(x)) \geq 0 \), we conclude that for any \( x \in X \)

\[ \rho(f(x), g(x)) = 0. \]

Since \( \rho \) is a metric, it implies that for any \( x \in X \) \( f(x) = g(x) \). Thus \( f = g \). The fact
that if \( f = g \) then \( d(f, g) = 0 \) is immediate:

\[ d(f, g) = \sup_{x \in X} \rho(f(x), g(x)) = \sup_{x \in X} 0 = 0. \]

b) property \( d(f, g) = d(g, f) \).

\[ d(f, g) = \sup_{x \in X} \rho(f(x), g(x)) = \sup_{x \in X} \rho(g(x), f(x)) = d(g, f). \]

b) property \( d(f, g) \leq d(f, h) + d(h, g) \).

\[ d(f, g) = \sup_{x \in X} \rho(f(x), g(x)) \leq \sup_{x \in X} (\rho(f(x), h(x)) + \rho(h(x), g(x))) \]

\[ \leq \sup_{x \in X} \rho(f(x), h(x)) + \sup_{x \in X} \rho(h(x), g(x)) = d(f, h) + d(h, g). \]

2. One direction. Suppose a topological space \( X \) is such that for any collection of closed subsets \( \mathcal{F} = F_\alpha | \alpha \in A \), that satisfies the finite intersection property we have

\[ \cap_{\alpha \in A} F_\alpha \neq \emptyset. \]

Let us assume, by contradiction, that \( X \) is not compact. Then there exists an open cover \( \mathcal{U} = U_\alpha | \alpha \in A \), that has no finite subcover. In other words we have

\[ X = \cup_{\alpha \in A} U_\alpha, \]

but for any \( U_1 \in \mathcal{U}, U_2 \in \mathcal{U}, \ldots, U_n \in \mathcal{U} \)

\[ X \cap (\cup_{i=1}^n U_i)^c \neq \emptyset. \]

Let us consider a family of closed sets

\[ \mathcal{F} = F_\alpha = U_\alpha^c | \alpha \in A. \]
Since we have that for any \( U_1 \in \mathcal{U}, U_2 \in \mathcal{U}, \ldots, U_n \in \mathcal{U} \)

\[
X \cap (\bigcup_{i=1}^{n} U_i)^c \neq \emptyset,
\]
it implies, by DeMorgan’s laws, the finite intersection property: for any finitely many \( F_1 \in \mathcal{F}, F_2 \in \mathcal{F}, \ldots, F_n \in \mathcal{F} \)

\[
\bigcap_{i=1}^{n} F_i \neq \emptyset.
\]

On the other hand

\[
X = \bigcup_{\alpha \in A} U_{\alpha},
\]
implies, again by DeMorgan’s laws

\[
\bigcap_{\alpha \in A} F_{\alpha} = \emptyset.
\]

Contradiction.

Other direction. Let us now assume that \( X \) is compact. Then for any open cover \( \mathcal{U} = U_\alpha | \alpha \in A \), that has a finite subcover. In other words whenever we have

\[
X = \bigcup_{\alpha \in A} U_{\alpha},
\]
then there are \( U_1 \in \mathcal{U}, U_2 \in \mathcal{U}, \ldots, U_n \in \mathcal{U} \) such that

\[
X = \bigcup_{i=1}^{n} U_i.
\]

Suppose, again by contradiction, that there is a collection of closed subsets \( \mathcal{F} = F_\alpha | \alpha \in A \), that satisfies the finite intersection property but

\[
\bigcap_{\alpha \in A} F_{\alpha} = \emptyset.
\]

Let us look at the complements of \( F_{\alpha} \).

\[
\mathcal{U} = U_{\alpha} = F_{\alpha}^c | \alpha \in A.
\]

Then by DeMorgan’s laws

\[
X = \bigcup_{\alpha \in A} U_{\alpha},
\]
but for any \( U_1 \in \mathcal{U}, U_2 \in \mathcal{U}, \ldots, U_n \in \mathcal{U} \) such that

\[
X \cap (\bigcup_{i=1}^{n} U_i)^c \neq \emptyset.
\]

Contradiction.

3. a) The first two properties are obvious. By definition, \( \rho \) is nonnegative and if words are different \( \rho > 0 \). If words \( w_1 \) and \( w_2 \) are the same then for any \( n \), their \( n \)th letters are the same and therefore for any \( n \)

\[
\rho(w_1, w_2) < \frac{1}{2^n}.
\]

Thus \( \rho(w_1, w_2) = 0 \). If the words \( w_1 \) and \( w_2 \) agree for the first \( n \) letters and are different at \((n+1)\)st letter, then again by definition

\[
\rho(w_1, w_2) = \frac{1}{2^n} = \rho(w_2, w_1).
\]

If a pair of words among \( w_1, w_2, w_3 \) is the same, then the triangle inequality is obvious. Suppose all \( w_1, w_2, w_3 \) are different. In order to prove the triangle inequality observe
that if $w_1$ and $w_2$ agree for the first $k$ letters and are different at the $(k + 1)$st letter, and $w_3$ and $w_2$ agree for the first $n$ letters and are different at the $(n + 1)$st letter, then $w_1$ and $w_3$ must agree for the first $m$ letters, where $m = \min(k, n)$. Hence

$$\rho(w_1, w_3) \leq \frac{1}{2m} = \max \left( \frac{1}{2n}, \frac{1}{2k} \right) < \frac{1}{2n} + \frac{1}{2k} = \rho(w_1, w_2) + \rho(w_2, w_3).$$

Note that if $w_1$, $w_2$, $w_3$ are different, then the triangle inequality is always a strict inequality.

b) If words $w_1$, $w_2$ and $w_3$ are listed in alphabetical order, it implies that if $w_1$ and $w_3$ agree in the first $n$ letters (and are different at the $(n + 1)$st letter), then $w_1$ and $w_2$ also must agree in the first $n$ letters. Therefore

$$\rho(w_1, w_2) \leq \frac{1}{2n} = \rho(w_1, w_3).$$

c)

$$\rho(w_1, w_3) = \max(\rho(w_1, w_2), \rho(w_2, w_3)).$$

In order to prove this formula, we will go back to our reasoning in part a) and observe that if $w_1$, $w_2$ and $w_3$ are listed in alphabetical order, it implies that $w_1$ and $w_3$ must agree for the first $m$ letters, where $m = \min(k, n)$, and they must be different at the $(m + 1)$st letter. Then

$$\rho(w_1, w_3) = \frac{1}{2m} = \max \left( \frac{1}{2n}, \frac{1}{2k} \right) = \max(\rho(w_1, w_2), \rho(w_2, w_3)).$$

4. Denote $\varepsilon_n = 1/n$. Let $X$ be a compact metric space. For any $n \in \mathbb{N}$, there exists an $\varepsilon_n$-net, that is points $x^n_1 \in X$, $x^n_2 \in X$, $\ldots$, $x^n_{k_n} \in X$ such that for any $x \in X$ there exists $j$, $1 \leq j \leq k_n$ such that $\rho(x, x^n_j) < \varepsilon_n$. Let

$$S_n = \{x^n_1, x^n_2, \ldots, x^n_{k_n}\}.$$ 

For each $n$, $S_n$ is a finite set, therefore

$$S = \bigcup_{n=1}^{\infty} S_n$$

is countable. We only need to verify that $S$ is dense in $X$. For any $\varepsilon > 0$, there exists $\varepsilon_n < \varepsilon$. For any $x \in X$, there exists $x^n_j \in S_n \subset S$ such that

$$\rho(x, x^n_j) < \varepsilon_n < \varepsilon.$$ 

Therefore $S$ is dense.

5. Let us denote, for convenience, $g(y) = f^{-1}(y)$. The $g(y)$ is a well-defined function

$$g : f(X) \to X, f(X) \subset Y,$$

because $f$ is bijective. In order to show that $g(y)$ is continuous we will use the following characterization of continuous functions: $g(y)$ is continuous if and only if for any $F \subset X$ closed set, $g^{-1}(F)$ is also closed.

Observe that for any $F \subset X$

$$g^{-1}(F) = f(F).$$
Therefore we need to show that if $F \subset X$ is closed then $f(F)$ is also closed. Here is a proof of this claim. If $F$ is closed, then $F$ is compact, because $X$ is compact, and $F \subset X$. $f(F)$ is compact in $Y$, because $f$ is continuous. $f(F)$ is closed because $f(F)$ is compact and $Y$ is a metric space.

In the first step we used a theorem that in a compact space every closed set is compact. In the last step we used a theorem that in a metric space every compact set is closed. In fact, in a Hausdorff space every compact set is closed (our space $Y$ is Hausdorff, because it is metric). Hence the problem can be reformulated as follows:

Prove that if $f$ is an bijective continuous function of a compact space $X$ onto a Hausdorff space $Y$, then $f^{-1}$ is continuous.

b) consider a map $f : [0, 1) \to \mathbb{R}^2$ given by

$$f : x \to (\cos 2\pi x, \sin 2\pi x).$$

$f$ is a continuous function, because $\cos 2\pi x$, $\sin 2\pi x$ are continuous.

$$f([0, 1)) = S_1 = \{y = (y_1, y_2) \in \mathbb{R}^2, \ y_1^2 + y_2^2 = 1\}.$$

So $f$ maps $[0, 1)$ onto a unit circle $S_1$. We know from Calculus, that $f$ is bijective. Hence we verified all the conditions of the part a), but the condition that $X = [0, 1)$ is compact. Clearly $X$ is not compact, because, if it were compact, it must be also a compact set of $\mathbb{R}$. But by the Heine-Borel theorem $[0, 1)$ is not compact in $\mathbb{R}$, because it is not closed. The map

$$f^{-1} : S_1 \to [0, 1)$$

is not continuous. Indeed, consider a point $y = (1, 0) \in \mathbb{R}^2$. Then for any $\varepsilon > 0$, we have

$$f([0, \frac{\varepsilon}{2\pi}] \cup (1 - \frac{\varepsilon}{2\pi}, 1)) \subset B_\varepsilon(y).$$

The last inequality follows from the a simple trigonometric calculation: for $x > 0$

$$\sqrt{\sin^2 2\pi x + (1 - \cos 2\pi x)^2} = 2\sin \frac{2\pi x}{2} \leq 2\pi x.$$

$$\sqrt{\sin^2 2\pi(1 - x) + (1 - \cos 2\pi(1 - x))^2} = 2\sin \frac{2\pi x}{2} \leq 2\pi x.$$  

Therefore

$$[0, \frac{\varepsilon}{2\pi}] \cup (1 - \frac{\varepsilon}{2\pi}, 1) \subset f^{-1}(B_\varepsilon(y) \cap S_1).$$

Thus choosing $\delta < 1/2$, for any $\varepsilon > 0$, there exists a point $\tilde{y} = (\cos \theta, -\sin \theta)$, $\tilde{y} \in B_\varepsilon(y) \cap S_1$, such that

$$|f^{-1}(y) - f^{-1}(\tilde{y})| = |1 - \frac{\theta}{2\pi}| > \delta$$

if we choose

$$\theta < \pi \text{ and } \theta < \varepsilon.$$