Solutions, hw 10

1. The triangle inequality is

\[ ||x - y||_2 \leq ||a - y||_2 + ||a - x||_2 \]

and

\[ ||a - y||_2 \leq ||x - y||_2 + ||a - x||_2 \]

From the first inequality we have:

\[ f_x(y) = ||x - y||_2 - ||a - y||_2 \leq ||a - x||_2, \]

from the second one:

\[ f_x(y) = ||x - y||_2 - ||a - y||_2 \leq -||a - x||_2. \]

Therefore

\[ |f_x(y)| \leq ||a - x||_2. \]

Since \( a \) and \( x \) are fixed, \( f_x(y) \) is bounded.

Now, let us prove that \( f_x(y) \) is continuous. Again, by triangle inequality for and \( h \in \mathbb{R}^n \)

\[ -||h||_2 \leq ||x - (y + h)||_2 - ||x - y||_2 \leq ||h||_2, -||h||_2 \leq ||a - (y + h)||_2 - ||a - y||_2 \leq ||h||_2. \]

Therefore

\[ f_x(y + h) - f_x(y) = ||x - (y + h)||_2 - ||a - (y + h)||_2 - ||x - y||_2 + ||a - y||_2 \\
\]

\[ = (||x - (y + h)||_2 - ||x - y||_2) - (||a - (y + h)||_2 - ||a - y||_2) \\
\]

\[ \leq ||x - (y + h)||_2 - ||x - y||_2 + ||a - (y + h)||_2 - ||a - y||_2 \leq 2||h||_2. \]

Similarly,

\[ f_x(y + h) - f_x(y) \geq -2||h||_2. \]

Hence

\[ |f_x(y + h) - f_x(y)| \leq 2||h||_2. \]

Therefore for any \( \varepsilon > 0 \), there exists \( \delta = \varepsilon/2 \), such that if

\[ z \in B_\delta(y) \Leftrightarrow ||z - y|| < \varepsilon/2, \]

we have, setting \( h = z - y \),

\[ |f_x(z) - f_x(y)| \leq 2||z - y||_2 < \varepsilon. \]

Finally,

\[ \rho(f_x, f_y) = \sup_{z \in \mathbb{R}^n} |f_x(z) - f_y(z)|. \]

But

\[ |f_x(z) - f_y(z)| = |||x - z||_2 - ||a - z||_2 - ||y - z||_2 + ||a - z||_2| \\
\]

\[ = |||x - z||_2 - ||y - z||_2| \]

Again, by triangle inequality

\[ -||x - y||_2 \leq ||x - z||_2 - ||y - z||_2 \leq ||x - y||_2. \]
and therefore
\[ \rho(f_x, f_y) \leq \|x - y\|_2. \]

We only need to prove that
\[ \sup_{z \in \mathbb{R}^n} |f_x(z) - f_y(z)| = \|x - y\|_2. \]

We can do it if we can find a point \( z \in \mathbb{R}^n \) such that
\[ |f_x(z) - f_y(z)| = \|x - y\|_2. \]

Using
\[ |f_x(z) - f_y(z)| = \||x - z||_2 - \|y - z||_2| \]
and choosing \( z = x \) we have
\[ |f_x(x) - f_y(x)| = \|x - y\|_2. \]

Hence
\[ \rho(f_x, f_y) = \|x - y\|_2. \]

2. Clearly, \( \rho(x, y) = |e^x - e^y| \geq 0, \rho(x, x) = |e^x - e^x| = 0. \) Suppose \( \rho(x, y) = 0, \) then
\[ e^x = e^y \Leftrightarrow e^{x-y} = 1 \Leftrightarrow x - y = 0. \]

Symmetry is also straightforward: \( \rho(x, y) = |e^x - e^y| = |e^y - e^x| = \rho(y, x). \) Finally, the triangle inequality:
\[ \rho(x, y) = |e^x - e^y| = |e^x - e^z + e^z - e^y| \leq |e^x - e^z| + |e^z - e^y| = \rho(x, z) + \rho(z, y). \]

3. Consider any point as a subset of \( X \) \( \{x\} \subset X. \) Let us show that it is open. Consider a neighborhood of \( x \) defined as
\[ U_x = \{y \in X | \rho(x, y) < 1/2\} \]

We have that \( U_x = \{x\}, \) and therefore for any \( y \in \{x\} \) (there is only one such \( y, \) namely \( y = x!\)) the set \( \{x\} \) also contains a neighborhood of \( y: U_x \subset \{x\}. \) Hence \( \{x\} \) is open. Consider any subset of \( X: S \subset X. \) Since
\[ S = \bigcup_{x \in S} \{x\}, \]
and each set \( \{x\} \) is open, we have that \( S \) is open. Therefore any subset of \( X \) is open. Finally, for any \( S \) consider \( S^c. \) Clearly \( S^c \subset X, \) and therefore it is open. Thus \( S \) is closed. Therefore any subset of \( X \) is closed.

4. Since we are in a metric space, we just need to show that the limit of any sequence in \( B_1(a), \) which converges to a point in \( X \) belongs to \( \tilde{B}_1(a). \) Suppose \( x_n \in B_1(a), n \in \mathbb{N}, \) and suppose \( x_n \to x, x \in X \) as \( n \to \infty. \) This means that
\[ \rho(x_n, a) < 1, \rho(x_n, x) \to 0. \]

We need to show that \( x \in \tilde{B}_1(a). \) This means that
\[ \rho(x, a) \leq 1. \]

By the triangle inequality we have for any \( n \)
\[ \rho(x, a) \leq \rho(x_n, a) + \rho(x_n, x) < 1 + \rho(x_n, x). \]
Therefore (by the limit theorems)

$$\rho(x, a) \leq 1 + \lim_{n \to \infty} \rho(x_n, x) = 1.$$ 

Hence

$$\overline{B}_1(a) \subset \tilde{B}_1(a).$$

The example, where the inclusion is proper is as follows. Consider the discrete metric from the previous exercise on $X$, that contains only two points: $X = \{0, 1\}$. Then

$$\overline{B}_1(0) = B_1(0) = \{0\}, \quad \tilde{B}_1(0) = X = \{0, 1\} \neq \overline{B}_1(0).$$