Review Continuity

1. Give an example of a sequence of real valued functions \( f_n(x, y) \) defined on \([0, 1] \times [0, 1]\) such that \( f_n \) converges pointwise, but not uniformly.

2. Let \( f_n \) be continuous functions such that \( f_n \rightarrow f \) uniformly on \([a, b]\).
   Prove that \( f_n^2 \rightarrow f^2 \) uniformly on \([a, b]\).

3. Let \( f_n \) be continuous functions such that \( f_n \rightarrow f \) uniformly on \([a, b]\).
   Prove that for any continuous function \( g(x) \)
   \[
   \int_a^b g(x)f_n(x)dx \rightarrow \int_a^b g(x)f(x)dx
   \]

4. A trigonometric series is a series of the form
   \[
   \sum_{n=0}^\infty (A_n \sin 2nx + B_n \cos 2nx).
   \]
   Prove that this series defines a continuous function, if
   \[
   \sum_{n=0}^\infty (|A_n| + |B_n|) < \infty.
   \]

5. Let \( K \subset X \) be a compact set, and \( F \subset C(K, \mathbb{R}) \) is some compact family of functions. Show that for every \( \phi: \mathbb{R} \rightarrow \mathbb{R} \) which is continuous on \( \mathbb{R} \), \( F_\phi = \{ \phi(f(x)) | f \in F \} \) is compact in \( C(K, \mathbb{R}) \).

6. Let \( K \subset X \) be a compact set. Suppose \( F \subset C(K, [0, 1]) \) is some compact family of functions. Prove that there exists \( f_0 \in F \) and \( x_0 \in K \) such that
   \[
   f_0(x_0) \geq \sup_{f \in F} \sup_{x \in K} f(x).
   \]