Topics and problems for review

There are eight topics to be covered on the final. The exam will have eight problems/questions about definitions/theorems.

1. Topology in $\mathbb{R}^n$, $l_p$, $C(X,Y)$, Chapter 6.1, notes. Especially topology in $C([0,1],\mathbb{R})$.
2. Continuous mappings Chapter 6.2, Chapter 3.4, notes.
4. Construction of Topological spaces Chapter 6.4 parts 6.26-6.28
5. Sequences in metric spaces Chapter 6.5 parts 6.34-6.41
6. Compactness Chapter 6.6
7. Connectedness Chapter 6.7
8. Approximation results for continuous functions Chapter 7.1 parts 7.1-7.3 and Chapter 3.3

Questions for review include all homework assignments and the following problems, loosely arranged by the above mentioned topics.

1. Topology. Definitions of open, closed, dense sets, convergence of sequences, compact sets etc.
   Identify which of the following is a closed, open, compact sets. Find their closure, interior, isolated points, limit points.

   $$\{f_n(x) = x^n \text{ on } [0,1]\}$$
   $$\{f(x) \in C([0,1],\mathbb{R}, \int_0^1 f(x) > 0\}$$
   $$\{f(x) \in C([0,1],\mathbb{R}, \int_0^1 f(x) = 0\}$$
   $$\{f(x) \in C([0,1],\mathbb{R}, |f(x) - f(y)| \leq |x - y|\}$$

2. Continuous mappings. Verify whether the following maps are continuous or not

   $$F : l_2 \to \mathbb{R}, F(x) = \sum_{n=1}^{\infty} f_n x_n,$$

where $\sum_{n=1}^{\infty} f_n^2 < \infty$

   $$F : C([0,1],\mathbb{R}) \to \mathbb{R}, F(f) = \max f - \min f$$
   $$F : S \to \mathbb{R}^2, F(x(a,q)) \to (a,q).$$

where

   $$S = \{x(a,q) \in l_2 ||x|| \leq 1, x = (a_x a_x q_x a_x q_x^2 a_x q_x^3 \ldots)\}$$

3. Metric spaces
   Give an example of a bounded, but not totally bounded metric space.
   Give an example of an metric space, which is not complete.
   Give an example of a compact metric space.

4. Construction of Topological spaces
   Give an example of sets $S \subset K \subset X$ which is closed in the relative topology of $K$, but it is not closed in topology of $X$.
   Give an example of sets $A \subset B$ such that $B$ is a complete metric space and $A$ is not
complete (in the relative topology).
Give an example of a base of a topological space.

5. Sequences in metric spaces
Verify whether the following sequence \( x_n \in X, x \) is metric is Cauchy

\[
\rho(x_{n+1}, x_n) \leq \frac{1}{2} \rho(x_n, x_{n-1}).
\]

6. Compactness
Consider a set \( F \subset C([0, 1], \mathbb{R}) \) with the properties
for any \( f \in F \) \( f \) is differentiable and \( |f'| \leq 1 \). Is \( F \) equicontinuous? Is \( F \) compact?

7. Connectedness
Show that the set \( X \subset \mathbb{R}^2 \)

\[
X = \{(t, \sin(1/t)) : t \neq 0\} \cup \{(0, t) : -1 \leq t \leq 1\}
\]
is connected

8. Approximations
Suppose that for a given function \( f(x) \in C([0, 1], \mathbb{R}) \) we have the following property: for each \( \varepsilon > 0 \) there exists a piecewise linear function \( g(x) \) such that

\[
d(f, g) < \varepsilon \text{ and } \int_0^1 g(x)dx = 0.
\]

Prove that \( \int_0^1 f(x)dx = 0 \)