

6.9. Poisson equation in a rectangle.

We consider

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad 0 < x < L, \quad 0 < y < H, \quad (1)$$

$$u(0, y) = u(L, y) = 0, \quad 0 < y < H, \quad (2)$$

$$u(x, 0) = u(x, H) = 0, \quad 0 < x < L. \quad (3)$$

We propose the eigenvalue problem

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\lambda u, & 0 < x < L, \quad 0 < y < H, \\ u(0, y) = u(L, y) = 0, & 0 < y < H, \\ u(x, 0) = u(x, H) = 0, & 0 < x < L. \end{cases} \quad (4)$$

We use separation of variables to find

$$u(x, y) = \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right), \quad \lambda = \left(\frac{n\pi x}{L}\right)^2 + \left(\frac{m\pi y}{H}\right)^2, \quad (5)$$

where $n = 1, 2, \dots, m = 1, 2, \dots$, are solutions to (4). Using Fourier sine series Theorem in the x -variable first and then in the y -variable, we can expand $f(x, y)$ into

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right), \quad (6)$$

where

$$B_{nm} = \frac{4}{LH} \int_0^L \int_0^H f(x, y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) dx dy. \quad (7)$$

Thus a solution to (1)-(3) is

$$u(x, y) = - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{B_{nm}}{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right). \quad (8)$$

where B_{nm} are given in (7). We can verify that (8) is indeed a solution:

$$\begin{aligned} \Delta u &= - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{B_{nm}}{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2} \Delta \left(\sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) \right) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) \\ &= f(x, y). \end{aligned}$$

Note: Let Ω be a general domain in \mathbb{R}^n . Consider

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (9)$$

Then in general there is no explicit formula for u or λ . But a general theorem tells us that there exists $\lambda_1 < \lambda_2 < \dots$ and corresponding u_1, u_2, \dots such that (λ_j, u_j) are solutions to the **eigenvalue problem** (9) and any $f(\vec{x})$ can be written as

$$f(\vec{x}) = \sum_{n=1}^{\infty} B_n u_n. \quad (10)$$

Thus a solution to

$$\Delta u = f(x)$$

is

$$u = - \sum_{n=1}^{\infty} \frac{B_n}{\lambda_n} u_n. \quad (11)$$

So the study of the eigenvalue problem is very useful. The expansion (10) is called the **eigen-function expansion**. The equation

$$-\Delta u = \lambda u \quad (12)$$

is called the **Helmholtz equation**.

The eigen functions u_n are orthogonal in the sense

$$\int_{\Omega} u_n(x) u_m(x) dx = 0, \quad \text{if } n \neq m. \quad (13)$$

This is why we can determine the coefficients B_n quickly:

$$B_n = \frac{\int_{\Omega} f(x) u_n(x) dx}{\int_{\Omega} (u_n(x))^2 dx} \quad (14)$$

for all $n = 1, 2, \dots$.