3.2. Bounded linear functionals, Riesz representation, and Dirac delta functional.

See Text by Keener, §3.1-2. pp.101–118.

Similar to an ordinary function $y = f(x)$ that is defined on the space $I\!R^n$ and takes on values in $I\!R^1$, a **functional** is a function on a function space B that determines uniquely a number in $I\!R^1$ for each member in B.

Example 1. Let $B = L^2[0,1]$ and

$$T_1 f = \int_0^1 f(x)dx.$$

Then T_1 is a functional on B.

A functional T is **linear** if it satisfies

$$T(\alpha f + \beta h) = \alpha T(f) + \beta T(h)$$

for all real numbers α and β and all members f and h in B.

We see the above T_1 is linear, but the following S

$$Sf = \int_0^1 (f(x))^2 dx$$

is not linear.

A functional T is **bounded** if there exists a number C such that

$$|Tf| \le C\|f\|_B$$

holds for all $f \in B$.

Example 2. Consider $B = L^2[0,1]$ and a $g(x) \in L^2[0,1]$. Define

$$T_g f = \int_0^1 g(x)f(x)dx.$$

We show that T_g is a bounded linear functional. Recall Cauchy-Schwarz inequality

$$|\int_0^1 fgdx| \le (\int_0^1 f^2 dx)^{1/2} (\int_0^1 g^2 dx)^{1/2}.$$

So $T_g f$ is a finite number for any $f \in B$. Thus T_g is defined on B as a functional. From the inequality we see that it is also bounded with the choice $C = \|g\|_{L^2}$. The linearity is obvious.

There are generally many bounded linear functionals. But there is an excellent representation of bounded linear functionals on Hilbert spaces.

Theorem (Riesz representation theorem) Any bounded linear functional T on a Hilbert space H can be represented by a member $g \in H$ in the form of the inner product

$$Tf = < g, f >, \quad \text{for all } f \in H.$$

Example 3. On $I\!R^n$, a linear function $f(x_1, x_2, \cdots, x_n)$ takes the form of an inner product with a vector $\alpha = (\alpha_1, \cdots, \alpha_n)$:

$$f(x_1, x_2, \cdots, x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n = \alpha \cdot x.$$

Let p be a number such that $1 \leq p < \infty$. And let q be such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We have

Theorem (Riesz representation part II) Any bounded linear functional T on $L^p[a,b]$ can be represented by a function $g \in L^q[a,b]$ in the form

$$Tf = \int_a^b f(x)g(x)dx.$$

We note that the Hölder inequality is helpful:

$$|\int_a^b f(x)g(x)dx| \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Dirac delta functional. Consider the Banach space $C[a,b]$. Its bounded linear functionals form a space called *finite Borel Measures*, which include the Dirac delta functional $\delta(x - x_0)$. Let us first consider the example functional:

$$Tf = f(x_0);$$

i.e., T takes any continuous function $f(x)$ to a number $f(x_0)$ where x_0 is a point in the interval $[a,b]$. This functional is linear since

$$T(\alpha f + \beta h) = \alpha f(x_0) + \beta h(x_0) = \alpha T(f) + \beta T(h).$$

It is bounded since

$$|Tf| = |f(x_0)| \leq \max_{x \in [a,b]} |f(x)| = \|f\|_{C^0}.$$

Traditionally this function is written as

$$Tf = \int_a^b \delta(x - x_0) f(x) dx \qquad (= f(x_0))$$

in line with the L^q representation of functionals on $L^p[a, b]$. In this representation, the $\delta(x - x_0)$ was regarded as a generalized function with the properties:

a. $\int_a^b \delta(x - x_0) dx = 1, (x_0 \in [a, b])$

b. $\delta(x - x_0) = 0$ for $x \neq x_0$.

We note that $\delta(x - x_0)$ is not a functional on the space $L^p[a, b]$ since an L^p function may not be defined on individual points.