1.10.3. The Deformation Tensor.

Let $\mathbf{u}(\mathbf{r})$ be the displacement of a point with position vector $\mathbf{r}$. Then the quantities

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

form a second-order tensor, called the deformation tensor.

1.10.3. The rate of Deformation Tensor.

Let $\mathbf{v}(M)$ be the velocity at a point M of a moving fluid. Then the quantities

$$v_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)$$

form a second-order tensor, called the rate of deformation tensor.

1.11. High-Order Tensors.

By a tensor of order n is meant a quantity uniquely specified by 3^n real numbers (the components of the tensor) which transform under changes of coordinate systems according to the law

$$A'_{i_1 i_2 \cdots i_n} = \alpha_{i'_1 k_1} \alpha_{i'_2 k_2} \cdots \alpha_{i'_n k_n} A_{k_1 k_2 \cdots k_n}$$

where $A_{k_1 k_2 \cdots k_n}$, $A'_{i_1 i_2 \cdots i_n}$ are the components of the vector in the old and new coordinate systems K and K' respectively, and $\alpha_{i'k}$ is the cosine of the angle between the i-th axis of K' and the k-th axis of K.

Example 1.11a. If $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ are three vectors, then the $3^3 = 27$ quantities

$$D_{ijk} = A_i B_j C_k$$

form a tensor of order 3. The proof is omitted, but see an exercise.

Example 1.11b. Suppose one second-order tensor A_{ik} is a linear function of another second-order tensor B_{ik}, such that

$$A_{ik} = \lambda_{iklm} B_{lm},$$

then λ_{iklm} form a fourth-order tensor. Proof is omitted.

1.12. Tensor Algebra.

1.12.1. Addition. We can add any two tensors of the same order, the sum is a tensor of the same order, whose components are the sums of the corresponding components of the two tensors. For example, tensor A_{ik} and tensor B_{ik} can be added to give a tensor C_{ik}:

$$C_{ik} = A_{ik} + B_{ik}.$$

1.12.2. Multiplication. We can multiply any number of tensors of arbitrary orders. The product of two tensors, for example, is a tensor whose order is the sum of the orders of the two tensors, and whose components are products of a component of one tensor with any component of the other tensor. The product of two second-order tensors A_{ik} with B_{lm}, for example, is a fourth-order tensor C_{iklm} with components

$$C_{iklm} = A_{ik}B_{lm}.$$

Our product of tensors is also called *outer product.*

1.12.3. Contraction of Tensors.

Summing a tensor of order n $(n \geq 2)$ over two of its indices is called contraction. For example, summing over the first and second indices of a third-order tensor

$$A_{iik} = A_{11k} + A_{22k} + A_{33k}$$

gives a vector. This is called contraction in the first and second indices. Contraction in both indices of a second-order tensor B_{ij} gives a scalar

$$B_{ii} = B_{11} + B_{22} + B_{33}.$$

Another example is A_{iki} gives another vector.

Contraction can be done many times.

Inner product. Multiplying two or more tensors and then contracting the product with respect to indices belonging to different factors is often called an *inner product* of the given tensors. For example, $A_{ik}B_k$, A_iB_i, and $\lambda_{iklm}B_{lm}$ are all inner products. But $A_{ii}B_k$ is not an inner product.