M598B: Mock Final Exam

Date: Dec. 4, 2001.

1. The stress tensor at a point has components given by

$$(s_{ij}) \quad = \quad \begin{pmatrix} 2 & -1 & 2 \\ -1 & 3 & 0 \\ 2 & 0 & -1 \end{pmatrix}.$$

Find the stress vector ($\mathbf{p}_n$) across an area normal to the unit vector

$$\mathbf{n} = (\mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3)/\sqrt{3}.$$

What is the normal stress across such an area (i.e, the projection $(\mathbf{p}_n \cdot \mathbf{n})\mathbf{n}$ of the vector $\mathbf{p}_n$ on to $\mathbf{n}$)?

2. Evaluate $\oint_{|z-1|=1} \frac{1}{(z-1)^2}\, dz$. The direction of the contour is counterclockwise.

3. Find the adjoint operator T^* for the operator T defined on the Hilbert space $L^2[a,b]$ by

$$Tu(x) = \int_a^b k(x)w(y)u(y)dy$$

where $k(x) \in L^2([a,b])$ and $w(y) \in C[a,b]$ are both given functions and u is an arbitrary member in $L^2[a,b]$.

4. Show that the Fourier transform of $f(x-c)$ is $e^{i\mu c}\hat{f}(\mu)$.

5. Determine whether the zero solution to the system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \epsilon\begin{pmatrix} y^2 \\ x^3 \end{pmatrix}$$

where ϵ is very small, is asymptotically stable or unstable. That is, determine whether all solutions with small data at $t = 0$ go to zero or at least one solution with arbitrarily small data fails to go to zero as time goes to plus infinity. (The size of ϵ depends on the size of the region of the initial data.)

6. Use the method of characteristics to derive a solution formula to

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + cu = 0, \qquad u(0,x) = g(x),$$

where a and c are constants, $t > 0$, and $x \in I\!R^1$.

7. Find the fundamental solution ϕ for a dipole source:

$$\Delta\phi(\mathbf{x}) = \nabla\delta(\mathbf{x})\cdot\mathbf{y} \equiv \lim_{h\to 0}\frac{\delta(\mathbf{x}+h\mathbf{y})-\delta(\mathbf{x})}{h},$$

where $\mathbf{y}$ is a fixed unit vector in $I\!R^3$. (In electrostatics, a dipole source is like a battery with its two terminals extremely close together in physical space.)

8. Find a solution to

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \qquad 0<x<L,\ 0<y<H\\
u &= 0 \quad \text{on the boundary of the rectangle,}\\
u(0,x,y) &= \sin\left(\frac{10\pi x}{L}\right)\sin\left(\frac{5\pi y}{H}\right)+\sin\left(\frac{20\pi x}{L}\right)\sin\left(\frac{4\pi y}{H}\right),\\
\frac{\partial u}{\partial t}(0,x,y) &= 0.
\end{aligned}$$