Supplemental Materials I. Linear Dependence

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1. Linear Dependence

Definition 1. A set of vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ is said to be *linearly dependent* if there exists a set of numbers $c_1, c_2, \cdots, c_n$, not all are zero, such that there holds

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{O}.$$

For example, the vectors

$$\mathbf{A} = (1, 0, 2), \qquad \mathbf{B} = (-2, 0, -4)$$

are linearly dependent since we can take $c_1 = 2, c_2 = 1$, and one of the cs is not zero.

Definition 2. A set of vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ is said to be *linearly independent* if the only solution to

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{O}$$

is the trivial one $c_1 = 0, c_2 = 0, \cdots, c_n = 0$.

For example, the set of vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ is linearly independent.

2. Basis. If a vector has m components, we say that it is an $m-$dimensional vector, or it is in the m-dimensional space R^m. Here m and n are positive integers.

Definition 3. If a set of n vectors in R^n is linearly independent, then it is called to form a **basis**, or simply it is a basis.

Theorem 1. In R^n, any vector $\mathbf{A}$ can be expressed uniquely as a linear combination of a set of basis. That is, let $\mathbf{A}$ be an arbitrary vector in R^n and $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ be linearly independent, then there exists a unique set of numbers $c_1, c_2, \cdots, c_n$ such that

$$\mathbf{A} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n.$$

To find the coefficients $c_1, c_2, \cdots, c_n$, one can solve the above system of equations.

Theorem 2. In the m dimensional space R^m, any $m + 1$ vectors or more are linearly dependent. This is because a homogeneous system of m linear equations with $m + 1$ or more unknowns always has nonzero solutions.