M597K: Solution to Homework Assignment 9

1. Solve the initial value problem for a first-order linear homogeneous equation

$$\frac{dx}{dt} - (\sin t)\, x = 0, (t > 0); \quad x(0) = 1.$$

Solution.

$$\frac{dx}{x} = (\sin t)\, dt$$

$$\int_0^t \frac{dx}{x} = \int_0^t \sin t\, dt$$

$$\ln x(t) - \ln x(0) = -\cos t|_0^t = 1 - \cos t$$

using $x(0) = 1$ we get

$$\ln x(t) = 1 - \cos t \quad \Rightarrow \quad x(t) = e^{1-\cos t}$$

2. Solve the initial value problem for a first-order linear nonhomogeneous equation

$$\frac{dx}{dt} + (t+1)x = e^{-t^2}, (t > 0); \quad x(0) = 0.$$

Solution.

$$\frac{dx}{dt} + a(t)x = f(t)$$

where

$$a(t) = t + 1, \quad f(t) = e^{-t^2}$$

using the formula

$$x(t) = e^{-\int_0^t a(s)ds}[x(0) + \int_0^t e^{\int_0^s a(\tau)d\tau} f(s)ds]$$

with $x(0) = 0$ to get

$$x(t) = e^{-\frac{t^2}{2}-t} \int_0^t e^{-\frac{s^2}{2}+s} ds$$

3. Find the general solution formula for the second-order linear scalar equation with constant coefficients

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0.$$

Solution.

We seek the solution in the form $x = e^{\lambda t}$, then

$$\frac{dx}{dt} = \lambda e^{\lambda t} \qquad \frac{d^2x}{dt^2} = \lambda^2 e^{\lambda t}$$

Plugging in the equation we get

$$(\lambda^2 + \lambda + 1)e^{\lambda t} = 0$$

Solve the characteristic equation

$$\lambda^2 + \lambda + 1 = 0$$

$$\Rightarrow \lambda_1 = \frac{-1 + \sqrt{3}i}{2}, \quad \lambda_2 = \frac{-1 - \sqrt{3}i}{2}$$

So the general solution is

$$x(t) = c_1 e^{\frac{-1+\sqrt{3}i}{2}t} + c_2 e^{\frac{-1-\sqrt{3}i}{2}t}$$

The real solution is

$$x(t) = e^{-\frac{1}{2}}[c_1 \cos(\frac{\sqrt{3}}{2}t) + c_2 \sin(\frac{\sqrt{3}}{2}t)]$$

4. Find the general solution formula for the first-order linear system of equations with constant coefficients

$$\begin{aligned} \tfrac{dx}{dt} &= 2x - y - z \\ \tfrac{dy}{dt} &= -x + 2y - z \\ \tfrac{dz}{dt} &= -x - y + 2z. \end{aligned}$$

Solution.

Let's seek the solution as:

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} ae^{\lambda t} \\ be^{\lambda t} \\ ce^{\lambda t} \end{bmatrix}$$

Then

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \lambda \begin{bmatrix} ae^{\lambda t} \\ be^{\lambda t} \\ ce^{\lambda t} \end{bmatrix} = \lambda \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

Plugging in to the equations we get $\lambda X = AX$.
That is $(\lambda I - A)X = 0$ where $X = (a, b, c)^T$
Solve the equation $det(\lambda I - A) = 0$ i.e.

$$\lambda^3 - 6\lambda^2 + 9\lambda = 0$$

To get the roots $\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 3$
For each λ_i, solve the equation $(\lambda_i I - A)X = 0$

$$\lambda_1 = 0 \quad \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \lambda_3 = 3 \quad \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{3t} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{3t}$$

5. Determine whether the zero solution to the system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \epsilon \begin{pmatrix} y^2 \\ x^3 \end{pmatrix}$$

where ϵ is very small, is asymptotically stable or unstable. That is, determine whether all solutions with small data at $t = 0$ go to zero or at least one solution with arbitrarily small data fails to go to zero as time goes to plus infinity. (The size of ϵ depends on the size of the region of the initial data.)

Solution.

Let

$$A = \begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} -3-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix}$$

Solve the equation $det(A - \lambda I) = 0$ i.e.

$$\lambda^2 + 5\lambda + 5 = 0$$

To get the roots

$$\lambda_{1,2} = \frac{-5 \pm \sqrt{5}}{2}$$

Because $\lambda_{1,2} < 0$, the system is asymptotically stable.

Optional 1. Find the general solution formula to the equation

$$\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x = 0.$$

Is the zero solution asymptotically stable?

Solution.

Let $x = e^{\lambda} \Rightarrow \lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

$\Rightarrow \lambda_{1,2,3} = -1 \quad$ triple roots

$\Rightarrow x(t) = (C_1 + C_2 t + C_3 t^2)e^{-t}$

Because $\lim_{t\to\infty} t^n e^{-t} = 0$ the solution is asymptotically stable.

Optional 2. Find the general solution formula to the system of equations

$$\frac{dx_1}{dt} = x_2 - x_3, \quad \frac{dx_2}{dt} = x_1 + x_2, \quad \frac{dx_3}{dt} = x_1 + x_3.$$

Solution.

Like the problem 4, to solve the equation $(\lambda I - A)X = 0$.

The roots are $\lambda_{1,2,3} = 0, 1, 1$

For $\lambda_1 = 0$ the eigenvector is $\alpha_1 = (1, -1, -1)^T$

For the double roots $\lambda_{2,3} = 1$, first we find a eigenvector $\alpha_2 = (0, 1, 1)^T$ and to solve the equation $(A - I)\alpha_3 = \alpha_2$ to get the $\alpha_3 = (1, 1, 0)^T$

so

$$\mathbf{X}(t) = C_1(1, -1, -1)^T + C_2(0, 1, 1)^T e^t + C_3(1, 1, 0)^T te^t$$