

## M597K: Solution to Homework Assignment 8

1. Follow the proof of Property b to prove

$$(f * g)^\vee(\mu) = \sqrt{2\pi} f^\vee(\mu) \cdot g^\vee(\mu).$$

*Solution.*

$$\begin{aligned}(f * g)^\vee(\mu) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\mu x} \left( \int_{-\infty}^{\infty} f(x-y)g(y)dy \right) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f(x-y)e^{-i\mu(x-y)}g(y)e^{-i\mu y}) dy dx\end{aligned}\tag{1}$$

Let  $x - y = z$

$$\begin{aligned}(f * g)^\vee(\mu) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f(z)e^{-i\mu z}) dz \int_{-\infty}^{\infty} g(y)e^{-i\mu y} dy \\ &= \sqrt{2\pi} f^\vee(\mu) \cdot g^\vee(\mu)\end{aligned}\tag{2}$$

2. Find the Fourier transform of

$$f(x) = \begin{cases} 0, & x < 0, \text{ or } x > a, \\ 1, & 0 \leq x \leq a. \end{cases}$$

*Solution.*

$$\begin{aligned}\hat{f}(\mu) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^1} f(x)e^{i\mu x} dx = \frac{1}{\sqrt{2\pi}} \int_0^a e^{i\mu x} dx \\ &= \frac{1}{\sqrt{2\pi}} \left( \frac{1}{i\mu} e^{i\mu x} \Big|_0^a \right) = \frac{e^{i\mu a} - 1}{i\sqrt{2\pi}\mu}\end{aligned}$$

3. Show that the Fourier transform of  $f(x)e^{icx}$  is  $\hat{f}(\mu + c)$ . Here  $c$  is a real number.

$$\begin{aligned}(\widehat{f(x)e^{icx}})(\mu) &= \int_{\mathbb{R}^1} e^{i\mu x} f(x)e^{icx} dx \\ &= \int_{\mathbb{R}^1} e^{i(\mu+c)x} f(x) dx = \hat{f}(\mu + c)\end{aligned}\tag{3}$$

4. Use Fourier transform to solve the heat (diffusion) equation with reaction:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + cu, \quad (x \in \mathbb{R}^1)$$

where  $k > 0$  and  $c$  is a real number, with initial condition

$$u(x, 0) = f(x).$$

*Solution.*

Note

$$\left(\frac{\partial u}{\partial t}\right)^\wedge = \frac{\partial \hat{u}}{\partial t}, \quad \left(\frac{\partial^2 u}{\partial x^2}\right)^\wedge = -\omega^2 \hat{u}$$

Fourier transform of original equations are:

$$\begin{cases} \frac{\partial \hat{u}}{\partial t} = (-k\omega^2 + c)\hat{u} \\ \hat{u}(0, \omega) = \hat{f}(\omega) \end{cases}$$

We get

$$\hat{u}(t, \omega) = \hat{f}(\omega)e^{(c-k\omega^2)t}$$

Let  $\hat{g}(\omega) = e^{(c-k\omega^2)t}$ ,

$$g(x) = [e^{(c-k\omega^2)t}]^\vee = [e^{ct-k\omega^2 t}]^\vee = e^{ct}[e^{-k\omega^2 t}]^\vee = \frac{1}{\sqrt{2kt}}e^{ct-\frac{x^2}{4kt}}$$

Then

$$\begin{aligned} u &= \frac{1}{\sqrt{2\pi}}f * g = \frac{1}{\sqrt{2\pi}} \int_{R^1} f(x-y) \frac{1}{\sqrt{2kt}} e^{ct-\frac{y^2}{4kt}} dy \\ &= \frac{1}{2\sqrt{\pi kt}} \int_{R^1} f(x-y) e^{ct-\frac{y^2}{4kt}} dy \end{aligned} \quad (4)$$

5. Find the Laplace transform of the function  $f(t) = t^2 + e^t, t \geq 0$ .

*Solution.*

$$L[f](s) = L[t^2 + e^t](s) = L[t^2](s) + L[e^t](s) = \frac{2!}{s^3} + \frac{1}{s-1} = \frac{2}{s^3} + \frac{1}{s-1}$$

6. Use Laplace transform to solve

$$\begin{aligned} u'' + 2u' + u &= e^{-t} \quad \text{for } t > 0, \\ u(0) &= 0, \\ u'(0) &= 0. \end{aligned}$$

*Solution.*

Let  $U = L[u]$

Then

$$\begin{aligned} L[u'] &= sL[u] - u(0) = sU \\ L[u''] &= s^2L[u] - su(0) - u'(0) = s^2U \\ L[e^{-t}] &= \frac{1}{s+1} \end{aligned}$$

Laplace transform of original equations is:

$$s^2U + 2sU + U = \frac{1}{s+1}$$

$$U = \frac{1}{(s+1)^3}$$

$$u = L^{-1}\left[\frac{1}{(s+1)^3}\right] = \frac{t^2}{2}e^{-t}$$

7. Use Laplace transform to solve

$$\begin{aligned}u'' + 2u' - 3u &= 0 & \text{for } t > 0, \\u(0) &= 1, \\u'(0) &= -2.\end{aligned}$$

*Solution.*

Let  $U = L[u]$

Then

$$L[u'] = sL[u] - u(0) = sU - 1$$

$$L[u''] = s^2L[u] - su(0) - u'(0) = s^2U - s + 2$$

Laplace transform of original equations is:

$$(s^2U - s + 2) + 2(sU - 1) - 3U = 0$$

$$U = \frac{s}{(s-1)(s+3)} = \frac{A}{s+3} + \frac{B}{s-1}$$

We have  $A + B = 1$ ,  $-A + 3B = 0$ .

Solve it we get  $B = \frac{1}{4}$ ,  $A = \frac{3}{4}$

$$U = \frac{3}{4(s+3)} + \frac{1}{4(s-1)}$$

$$u(t) = L^{-1}\frac{3}{4(s+3)} + L^{-1}\frac{1}{4(s-1)} = (3e^{-3t} + e^t)/4$$

(Seven problems in all, about 14.4 points each.)