

## M597K: Solution to Homework Assignment 7

The following problems are on the specified pages of the text book by Keener (2nd Edition, i.e., revised and updated version)

Problems 3 and 4 of Section 2.1 on p.94;

Problem 1 of Section 3.1 on p.128;

Problem 1 of Section 3.2 on p.128.

1. Show that the sequence  $(x_n)$ ,  $x_n = \sum_{k=1}^n \frac{1}{k!}$  is a Cauchy sequence using the measure of distance  $d(x, y) = |x - y|$ .

*Solution.*

A sequence of points  $x_n \in \mathbb{R}$  is Cauchy if for every  $\epsilon > 0$  there exists an  $N$ , such that  $m > n > N$  implies  $|x_n - x_m| < \epsilon$

Let  $\epsilon > 0$  be given, choose  $N = \frac{1}{\sqrt{\epsilon}}$ . Then for  $m > n > N$ , we have

$$\begin{aligned} |x_n - x_m| &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots + \frac{1}{m!} \\ &\leq \frac{1}{n(n+1)} + \frac{1}{n(n+1)(n+2)} + \dots \\ &\leq \frac{1}{n(n+1)} + \frac{1}{n(n+1)(n+1)} + \dots \\ &\leq \frac{1}{n(n+1)} \left( 1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots \right) \\ &\leq \frac{1}{n(n+1)} \cdot \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{n^2} = \epsilon. \end{aligned}$$

Therefore the sequence is Cauchy.

2. Show that the continuous functions on  $[0, 1]$  form an infinite dimensional vector space. Find a set of linearly independent vectors which is not finite.

*Solution.*

To prove the continuous functions on  $[0, 1]$  form a vector space is trivial, omitted.

We show that the polynomials are already infinite dimensional. Functions  $1, x, x^2, x^3, \dots$  is infinite and none is a linear combination of the others. They are linearly independent because the only way we can write  $c_1 x^n + c_2 x^m + c_3 x^l + \dots = 0$  (assuming  $n > m > l > \dots$ ) is if all the coefficients are zero. Because otherwise we will have a polynomial of a finite degree  $n$  which has at most  $n$  roots.

Thus the continuous functions form an infinite dimensional vector space. and  $1, x, x^2, x^3, \dots$  are independent.

3. Verify that the solution of  $u'' = f(x)$ ,  $u(0) = u(1) = 0$  is given by

$$u(x) = \int_0^1 k(x, y) f(y) dy$$

where

$$k(x, y) = \begin{cases} y(x-1) & , 0 \leq y < x \leq 1, \\ x(y-1) & , 0 \leq x < y \leq 1. \end{cases}$$

*Solution.*

We should first check the boundary:

$$u(0) = \int k(0, y) f(y) dy = 0$$

$$u(1) = \int k(1, y) f(y) dy = 0$$

And

$$\begin{aligned} u(x) &= \int_0^1 k(x, y) f(y) dy \\ &= \int_0^x y(x-1) f(y) dy + \int_x^1 x(y-1) f(y) dy \\ &= \int_0^1 xy f(y) dy - \int_0^x y f(y) dy - \int_x^1 x f(y) dy \\ u'(x) &= \int_0^1 y f(y) dy - x f(x) + \int_1^x f(y) dy + x f(x) = \int_1^x f(y) dy \\ u''(x) &= f(x) \end{aligned}$$

4. Show  $T(f) = f(0)$ , defined for all continuous functions on  $[-1, 1]$ , is not a bounded linear functional in the  $L^2$  norm, but it is a bounded linear functional with the uniform norm.

*Solution.*

For  $T$  to be bounded on continuous functions in the  $L^2$  norm, there must exist a constant  $C$  such that

$$|Tf| \leq C \|f\|_{L^2}, \text{ for all continuous } f(x).$$

The converse: i.e., for  $T$  not bounded, the statement is: For any finite number  $C$ , there is always a continuous function  $f$ , which may depend on  $C$ , thus we use  $f_C(x)$  to denote it, such that

$$|Tf| > C \|f\|_{L^2}. \tag{1}$$

We prove that such an  $f_C(x)$  exists. So let  $C$  be a positive number. Take  $f_C(x)$  be such that

$$f_C(x) = C, \quad x \in [-d, d],$$

and  $f_C(x) = 0$  for all other  $x$ , where  $d$  is a small number. Thus, we have  $|Tf_C(x)| = |f_C(0)| = C$ , and  $\|f_C(x)\|_{L^2} = C\sqrt{2d}$ . If we take  $d = \frac{1}{8C^2}$ , then inequality (1) holds like  $C > C \cdot \frac{1}{2}$ . Thus this  $T$  is not a bounded linear functional on the set of continuous functions measured in the norm  $L^2$ . (The function  $f_C(x)$  can be made continuous easily without changing its primary property.)

Since  $|f(0)| \leq \max |f(x)|$ ,  $T$  is bounded with uniform norm (take  $C = 1$ ).

**5.** Let  $\ell^2$  denote all the sequences  $(x_1, x_2, x_3, \dots, x_n, \dots)$  of real numbers. Let  $x$  denote such a sequence. Use vector addition and scalar multiplication. Then it is a vector space (no proof needed). Use the inner product

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$$

where  $y = (y_1, y_2, \dots, y_n, \dots)$ . Show that this inner product is well defined in  $\ell^2$  and it satisfies the four properties of the definition of inner product. (It is called “the little  $\ell$  two space”. (Reference: p. 59 of text book)

*Solution.*

By Cauchy-Schwarz inequality:

$$|\langle x, y \rangle| = \left| \sum_{i=1}^{\infty} x_i y_i \right| \leq \sqrt{\sum_{i=1}^{\infty} x_i^2} \sqrt{\sum_{i=1}^{\infty} y_i^2} = \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}}$$

Thus  $|\langle x, y \rangle| < \infty$ ,  $\langle x, y \rangle$  is well defined in  $\ell^2$ .

And

$$(1) \langle x + y, z \rangle = \sum_{i=1}^{\infty} (x_i + y_i) z_i = \sum_{i=1}^{\infty} x_i z_i + \sum_{i=1}^{\infty} y_i z_i = \langle x, z \rangle + \langle y, z \rangle$$

$$(2) \langle \alpha x, z \rangle = \alpha \sum_{i=1}^{\infty} x_i z_i = \alpha \langle x, z \rangle.$$

$$(3) \langle x, y \rangle = \langle y, x \rangle$$

$$(4) \langle x, x \rangle \geq 0 \text{ and } \langle x, x \rangle = 0 \text{ iff } x = 0$$

This  $\langle \cdot, \cdot \rangle$  is an inner product.

**6.** Let  $f(t)$  be equal to 100 for  $t$  between 19 and 20, and equal to zero for all other times  $t$ . This may represent a Miss Universe’s status score history of her lifetime.

Let  $g(t)$  be equal to 85 for  $t$  between 35 and 116, and equal to zero for all other times  $t$ . This might represent another person's social status score history, who at age 35 invented a perpetual machine and enjoyed the fame he got throughout his lifetime. (Unfortunately he lived only to 116. Thus his perpetual machine was just somewhat perpetual, and that explains the score 85 instead of a higher number.) Now a panel want to select one from the two to put into a Hall of Fame and want you to give the panel a single measurement number from each of the two so that the panel can decide. Do the maximum norm and the  $L^2$  norm calculation (serious part) and decide which norm you want to use to give to the panel (a decision that is purely up to you).

*Solution.*

$$f(t) = \begin{cases} 100 & , \quad 19 \leq t \leq 20, \\ 0 & , \quad \text{else,} \end{cases}$$

$$g(t) = \begin{cases} 85 & , \quad 35 \leq t \leq 116, \\ 0 & , \quad \text{else,} \end{cases}$$

$$\| f(t) \|_{\infty} = 100, \quad \| g(t) \|_{\infty} = 85$$

$$\| f(t) \|_{L^2} = (\int_{19}^{20} 100^2 dx)^{\frac{1}{2}} = 100, \quad \| g(t) \|_{L^2} = (\int_{35}^{116} 85^2 dx)^{\frac{1}{2}} = 765$$

7. Solve Problem 5, p. 94, of the text book, where  $L^2$  is replaced by  $L^1$ .

*Solution.*

As  $f_n(t) = 0$  when  $0 \leq t < \frac{1}{2} - \frac{1}{n}$ , and  $f_n(t) = 1$  when  $t > \frac{1}{2} + \frac{1}{n}$

Suppose  $n < m$ ,  $f_n(t) - f_m(t) = 0$  when  $0 \leq t < \frac{1}{2} - \frac{1}{n}$  or  $t > \frac{1}{2} + \frac{1}{n}$  and  $|f_n(t) - f_m(t)| \leq |f_n(t)| + |f_m(t)| \leq 2$

Thus  $\| f_n(t) - f_m(t) \|_{L^2} = (\int_0^1 (f_n(t) - f_m(t))^2)^{\frac{1}{2}} \leq (\int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}+\frac{1}{n}} (f_n(t) - f_m(t))^2)^{\frac{1}{2}} \leq (\int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}+\frac{1}{n}} (2)^2)^{\frac{1}{2}} = \sqrt{\frac{8}{n}} \rightarrow 0$  as  $n \rightarrow \infty$

That implies it is Cauchy in  $L^1$

8. Show that the functional  $T$  on the Banach space  $C[0, 1]$  defined by

$$Tf = \int_0^1 \frac{f(x)}{\sqrt{x}} dx, \quad f(x) \in C[0, 1]$$

is linear and bounded.

*Solution.*

$$T(\alpha f + \beta g) = \int_0^1 \frac{\alpha f(x) + \beta g(x)}{\sqrt{x}} dx = \alpha \int_0^1 \frac{f(x)}{\sqrt{x}} dx + \beta \int_0^1 \frac{g(x)}{\sqrt{x}} dx = \alpha T(f) + \beta T(g).$$

Thus  $T$  is linear.

$$\text{And } |Tf| = \left| \int_0^1 \frac{f(x)}{\sqrt{x}} dx \right| \leq \max |f(x)| \int_0^1 \frac{1}{\sqrt{x}} dx = 2 \cdot \max |f(x)|$$

Thus  $T$  is bounded.

**9.** Let  $a$  and  $b$  be two points in the interval  $[0, 1]$ . Show that the functional  $T$  on the Banach space  $C[0, 1]$  defined by

$$Tf = f(a) - f(b), \quad f(x) \in C[0, 1]$$

is linear and bounded. (This functional is generally written as  $\delta(x - a) - \delta(x - b)$ .)  
(Hint: Lecture notes might help.)

*Solution.*

$$T(\alpha f + \beta g) = \alpha(f(a) - f(b)) + \beta(g(a) - g(b)) = \alpha T(f) + \beta T(g).$$

Thus  $T$  is linear.

$$\text{And } |Tf| = |f(a) - f(b)| \leq |f(a)| + |f(b)| \leq 2 \cdot \max |f(x)|$$

Thus  $T$  is bounded.

**10.** Find the adjoint operator  $T^*$  for the operator  $T$  defined on the Hilbert space  $L^2[a, b]$  by

$$Tu(x) = w(x) \int_a^b k(x, y)u(y)dy$$

where  $k(x, y) \in L^2([a, b] \times [a, b])$  and  $w(x) \in C[a, b]$  are both given functions and  $u$  is an arbitrary member in  $L^2[a, b]$ . (Hint: Read text book p.107.)

*Solution.* Starting with the definition of  $Tu(x) = w(x) \int_a^b k(x, y)u(y)dy$  and the definition of  $T^*$  we formulate

$$\begin{aligned} \langle Tu, v \rangle &= \int_a^b (w(x) \int_a^b k(x, y)u(y)dy)v(x)dx \\ &= \int_a^b \int_a^b w(x)k(x, y)v(x)u(y) dx dy \\ &= \int_a^b (\int_a^b k(x, y)w(x)v(x) dx)u(y) dy \\ &= \langle T^*v, u \rangle . \end{aligned}$$

Thus  $T^*v(y) = \int_a^b k(x, y)w(x)v(x)dx$  or equivalently but more standard:

$$T^*u(x) = \int_a^b k(y, x)w(y)u(y) dy.$$

11. Verify that  $\lambda = (n\pi)^2$  and  $u = \cos(n\pi x)$  are eigenvalues and corresponding eigenfunctions for the Sturm-Liouville eigenvalue problem:

$$\frac{d^2u}{dx^2} + \lambda u = 0, (0 < x < 1); \quad u'(0) = u'(1) = 0$$

for all positive integer  $n$ .

*Solution.*

From  $\lambda = (n\pi)^2$  and  $u = \cos(n\pi x)$ , we can get  $\frac{du}{dx} = -n\pi \sin(n\pi x)$ , and

$$\frac{d^2u}{dx^2} + \lambda u = 0, (0 < x < 1)$$

And as  $u'(x) = -n\pi \sin(n\pi x)$ ,  $u'(0) = 0$ ,  $u'(1) = 0$ .