M597K: Hints to Homework Assignment 7

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1. To show a sequence $\{x_n\}$ is a Cauchy sequence, one needs to form the difference $|x_n - x_m|$ and *estimate* the difference. By using a series of inequalities one can arrive at

$$|x_n - x_m| < \frac{1}{n^2}$$

for all $m > n$. See my assistant Xiaoqiang Wang in his office at 217 Osmond Building if you have difficulties in that derivation. His e-mail address is fullheart@etang.com

Then let N be such that $\frac{1}{N^2} = \epsilon$, i.e., $N = \frac{1}{\sqrt{\epsilon}}$. Then for any $\epsilon > 0$, choose $N = \frac{1}{\sqrt{\epsilon}}$. Then for any $m > n > N$, there holds

$$|x_n - x_m| < \epsilon.$$

For the technical inequality, one can use

$$\begin{aligned}
&\tfrac{1}{(n+1)!} + \tfrac{1}{(n+2)!} + \cdots + \tfrac{1}{m!}\\
&\le \tfrac{1}{n(n+1)} + \tfrac{1}{n(n+1)(n+2)} + \cdots + \text{ to } \infty\\
&\le \tfrac{1}{n(n+1)} + \tfrac{1}{n(n+1)(n+1)} + \cdots + \text{ to } \infty\\
&\le \tfrac{1}{n(n+1)}(1 + \tfrac{1}{n+1} + \tfrac{1}{(n+1)^2} + \cdots)\\
&\ \text{use sum of geometric series.}\\
&\le \tfrac{1}{n(n+1)} \cdot \tfrac{1}{1-\frac{1}{n+1}}\\
&= \tfrac{1}{n^2}.
\end{aligned}$$

There are many other ways to estimate the above sum.

2. Use the set of functions $1, x, x^2, x^3, \cdots$. It is infinitely long. None of them is a linear combination of a finite number of others. Remember that two functions $f(x)$ and $g(x)$ are equal only if they are identical on $[0, 1]$. Even if they are equal on a finite number of points, they are not the same function provided that they differ on at least one point of the domain.

Also remember that a polynomial of degree n has at most n roots.

3. No hint.

4. For the uniform case, the proof is straightforward.

For the L^2 norm, use a sequence of functions that approximate the delta function δ.

5. Use the Cauchy-Schwarz inequality:

$$|\sum_{i=1}^{\infty} x_i y_i| \leq \sqrt{\sum_{i=1}^{\infty} x_i^2}\sqrt{\sum_{i=1}^{\infty} y_i^2}$$

for showing the well defined-ness (i. e., $| < x, y > | \ < \infty$).

6. No hint.

7. The following example might help understand L^1 convergence. This convergence involves the smallness of the domain on which the functions might not vanish.

Example 1. Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges to the zero function $f(x) = 0$, where

$$f_n(x) = 1 \quad \text{for } x \in (0, \frac{1}{n}); \qquad f_n(x) = 0 \text{ for all other } x$$

in the norm $L^1[-1, 1]$.

Proof. We calculate the norm

$$\|f_n - f\|_{L^1[-1,1]} = \int_{-1}^{1} |f_n(x) - f(x)|\, dx = \int_{-1}^{1} f_n(x)\, dx = \int_0^{\frac{1}{n}} 1\, dx = \frac{1}{n}$$

which is smaller than any given $\epsilon > 0$ for all $n > N \equiv \frac{1}{\epsilon}$.

Back to our homework problem. To show it is not a Cauchy sequence in $C[0, 1]$, one needs to find a number ϵ_0 and a pair of sequences f_n and f_m such that

$$\|f_n - f_m\|_{C^0} > \epsilon_0$$

I suggest to use $m = 2n$ and calculate the difference $f_n(t) - f_m(t)$ at the point $t = \frac{1}{2} - \frac{1}{m}$. It should be $\frac{1}{4}$. So take $\epsilon_0 = \frac{1}{8}$. Then for this ϵ_0, for the entire pair f_n and f_{2n} the distance of f_n and f_{2n} measaured in the maximum norm is at least $\frac{1}{4}$ for all $n \to \infty$.

8. Use definition, the maximum norm on f, and $\frac{1}{\sqrt{x}}$ is integrable.

9. It is similar to the functional $\delta(x)$. For linearity verify that

$$T(\alpha f(x) + \beta g(x)) = \alpha T f(x) + \beta T g(x).$$

Review the lecture note on boundedness.

10. Review the example in the text book, replace its $k(x, y)$ by $k(x, y)w(x)$.

11. Direct plugging.