M597K: Solutioin to Homework Assignment 6

Date: October 2, Wednesday, 2002. Due Friday. Oct. 11.

1. Find the numerical value of each of the following and express it in the form $a+bi$ where a and b are real numbers.

(a) $i(1+i)(2+i)$,

(b) $(1-i)^2+(3-2i)^2$,

(c) i^{2002}.

Solution.

(a) $i(1+i)(2+i) = i(2+3i-1) = i(1+3i) = i-3 = -3+i$,

(b) $(1-i)^2+(3-2i)^2 = 1-2i+i^2+9-12i+4i^2 = 5-14i$,

(c) $i^{2002} = (i^2)^{1001} = (-1)^{1001} = -1$.

2. Find the polar representation ($re^{i\theta}$) for each of the following

(a) $1-i$,

(b) $(1+i)/(1+\sqrt{3}i)$.

Solution.

(a) $1-i = \sqrt{2}e^{-\frac{\pi}{4}i}$,

(b) $(1+i)/(1+\sqrt{3}i) = \frac{\sqrt{2}e^{\frac{\pi}{4}i}}{2e^{\frac{\pi}{3}i}} = \frac{\sqrt{2}}{2}e^{-\frac{\pi}{12}i}$.

3a. From $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$, one can derive

$$\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}), \quad \sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}).$$

Then define

$$\cos z = \frac{1}{2}(e^{iz}+e^{-iz}), \quad \sin z = \frac{1}{2i}(e^{iz}-e^{-iz})$$

for all complex z. Find the derivatives $\cos' z$ and $\sin' z$.

Solution.

$\cos' z = \frac{1}{2}(e^{iz}+e^{-iz})' = \frac{i}{2}(e^{iz}-e^{-iz}) = -\frac{1}{2i}(e^{iz}-e^{-iz}) = -\sin z$

Same as above, we can get $\sin' z = \cos z$

3b. From $\sinh\theta = \frac{1}{2}(e^{\theta}-e^{-\theta})$ and $\cosh\theta = \frac{1}{2}(e^{\theta}+e^{-\theta})$, we define $\sinh z$ and $\cosh z$ for all complex z by replacing the real θ by z. Find the derivatives $\sinh'' z$ and $\cosh'' z$.

Solution.

Same as **3a.**, $\sinh' z = \cosh z$, $\cosh' z = \sinh z$, then $\sinh'' z = \cosh' z = \sinh z$, $\cosh'' z = \sinh' z = \cosh z$

4. Describe the range of the function

$$w = \ln z = \ln r + i\theta \qquad \text{for } z = re^{i\theta}, \quad \theta \in (-\pi, \pi)$$

with the domain $\Omega = \{z = re^{i\theta} \mid r > 0, -\pi < \theta < \pi\}$. What is the image of a circle $|z| = c > 0$? What is the image of a ray $\theta = \alpha$?

Solution.

The range is the strip: $-\infty < x < \infty, -\pi < y < \pi$.

The circle $|z| = c$ is taken to the vertical line segment: $x = \ln c, -\pi < y < \pi$

The ray $\theta = \alpha$ is taken to be a horizontal line: $-\infty < x < \infty, y = \alpha$

The map from z to ω seems to do the following:

See separate jpg file of the figure

5. Use the Cauchy-Riemann conditions to verify that the function

$$f(z) = \frac{1}{2}\ln(x^2 + y^2) + i \arctan \frac{y}{x},$$

where $z = x + iy, -\pi < \arctan \frac{y}{x} \leq \pi$ is analytic in the right half plane $x > 0$.

Solution.

Check the Cauchy-Riemann conditon:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

From f, $u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$, $v(x, y) = \arctan \frac{y}{x}$, checking is omitted.

6. Is $f(z) = \bar{z}z$ (where bar denotes complex conjugate) analytic in the entire plane?

Solution.

$$f = (x - iy)(x + iy) = x^2 + y^2$$

$$u = x^2 + y^2, v = 0$$

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial v}{\partial y} = 0$$

Thus, $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$, so f is not analytic.

7. Find the Taylor series for $f(z) = \frac{1+z}{z^2-1}$ at $z_0 = 0$.

Solution.

$$f(z) = \frac{1+z}{z^2-1} = \frac{1}{z-1} = -\frac{1}{1-z} = -(1+z+z^2+z^3+...)$$

8. Find the complex integral $\int_C \bar{z}\,dz$ where C is the arc $y = x^3$ for $0 \leq x \leq 1$. The direction of integration is from $x = 0$ to $x = 1$.

Solution.

$$\begin{aligned}\int_c \bar{z}\,dz = \int_c (x-iy)(dx+idy) \quad &= \textstyle\int_0^1 (x-ix^3)(dx+idx^3) \\ &= \textstyle\int_0^1 (x-ix^3)(1+3x^2 i)\,dx \\ &= \textstyle\int_0^1 x - ix^3 + 3x^3 i + 3x^5\,dx \\ &= (\tfrac{1}{2}x^2 - i\tfrac{x^4}{4} + \tfrac{3}{4}x^4 i + \tfrac{3}{6}x^6)|_0^1 \\ &= 1 + \tfrac{1}{2}i\end{aligned}$$

9. Find $\int_C (z^n + z + 1)\,dz$ where $n \geq 1$ is an integer and C is a closed contour.

Solution. The answer is zero since $z^n + z + 1$ is analytic and c is closed.

10. Evaluate $\oint_{|z-1|=1} \frac{z+1}{(z-1)^2}\,dz$. The direction of the contour is counterclockwise. Note that the symbol $|z-1| = 1$ denotes the circle

$$(x-1)^2 + y^2 = 1.$$

Solution. Use form

$$\int_c \frac{f(z)}{(z-z_0)^2}\,dz = 2\pi i f'(z_0)$$

so

$$\int_c \frac{z+1}{(z-1)^2}\,dz = 2\pi i$$