M597K: Solutioin to Homework 5

Date: Friday, October 4.

1. Recall that the product of two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ is defined as the matrix $AB = (c_{ij})$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad (i, j = 1, 2, \cdots, n).$$

Thus show that

$$\begin{bmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} & \frac{\partial x}{\partial u_3} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} & \frac{\partial y}{\partial u_3} \\ \frac{\partial z}{\partial u_1} & \frac{\partial z}{\partial u_2} & \frac{\partial z}{\partial u_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1}$$

Here (x, y, z) represents cartesian coordinates and (u_1, u_2, u_3) represents curvilinear coordinates whose Jacobian is not zero. (From this equation, and the rule

$$\det\,(AB) = \det\,(A)\,\det(B),$$

one can easily deduce that the Jacobian of the inverse transformation is the reciprocal of the Jacobian of the (forward) transformation: i.e., identity (4) in Section 1.15, Lecture 12.)

Solution. (10 points for each problem) We know that

$$\frac{\partial u_k}{\partial u_j} = \delta_{kj}$$

because u_1, u_2, and u_3 are independent variables. Using the chain rule, we obtain that

$$\frac{\partial u_k}{\partial u_j} = \frac{\partial u_k}{\partial x_i}\frac{\partial x_i}{\partial u_j} = \frac{\partial u_k}{\partial x_1}\frac{\partial x_1}{\partial u_j} + \frac{\partial u_k}{\partial x_2}\frac{\partial x_2}{\partial u_j} + \frac{\partial u_k}{\partial x_3}\frac{\partial x_3}{\partial u_j}.$$

Thus

$$\frac{\partial u_k}{\partial x_i}\frac{\partial x_i}{\partial u_j} = \delta_{kj},$$

which is the same as (1).

2. The transformation relating the cartesian coordinates x, y, z to the elliptic cylindrical coordinates u, v, z is given by the equations

$$x = a \cosh u \cos v, \quad y = a \sinh u \sin v, \quad z = z$$

($u \geq 0, 0 \leq v < 2\pi, a > 0$ constant).

(a) Show that in the xy-plane a curve $u =$ constant represents an ellipse, while a curve $v =$ constant represents half of one branch of a hyperbola.

Solution. Let $u = b$. In the xy-plane, we have

$$x = a \cosh b \cos v, \quad y = a \sinh b \sin v.$$

It can be written as

$$(x/(a \cosh b))^2 + (y/(a \sinh b))^2 = 1,$$

which represents an ellipse. It occupies the whole ellipse since both $\cosh u$ and $\sinh u$ are nonnegative, and v goes through the entire circle $[0, 2\pi)$.

Let $v = c$. Using

$$\cosh^2 u - \sinh^2 u = ((e^u + e^{-u})/2)^2 - ((e^u - e^{-u})/2)^2 = 1,$$

we find

$$(x/(a \cos c))^2 - (y/(a \sin c))^2 = 1.$$

This is a hyperbola. It has two branches; one branch passes through the point $(a \cos c, 0)$ while the other passes through the point $(-a \cos c, 0)$. For $c \in (0, \pi/4)$, we see that both x and y are positive, thus it lies in the first quadrant, and $c =$ constant represents the half branch in the first quadrant. Similarly, one can find that for $v \in (\pi/2, \pi)$, the branch is in the second quadrant. And so on.

(b) Sketch each curve on the xy-plane corresponding to the values $u = 0$; $v = 0$; $v = \pi$; $v = \pi/2$; respectively.

Solution. $u = 0$ is the segment $y = 0, -a < x < a$.

$v = 0$ is the ray $y = 0, x > a$.

$v = \pi$ is the ray $y = 0, x < -a$.

$v = \pi/2$ is the ray $x = 0, y > 0$.

See figure.

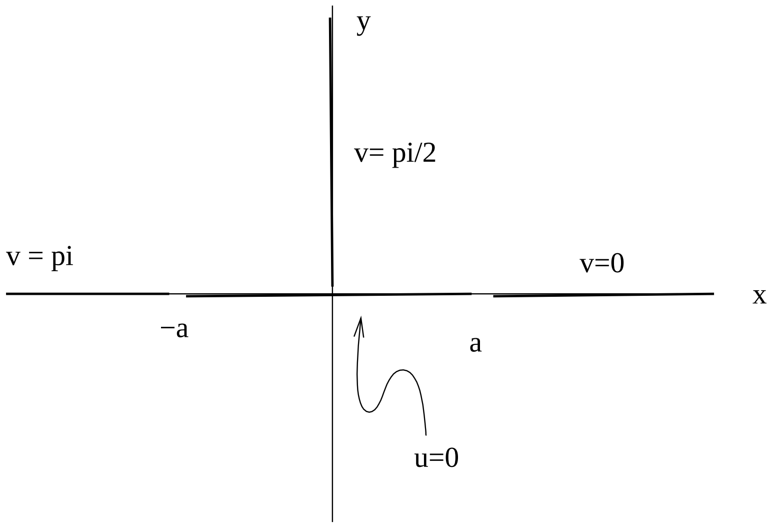

Figure Homework 5

(c) Verify that the new coordinate system is orthogonal.

Solution. Let $\mathbf{R} = x_i\mathbf{i}_i = a\cosh u \cos v\mathbf{i}_1 + a\sinh u \sin v\mathbf{i}_2 + z\mathbf{i}_3$. Then

$$\frac{\partial \mathbf{R}}{\partial u} = a\sinh u \cos v\mathbf{i}_1 + a\cosh u \sin v\mathbf{i}_2, \quad \frac{\partial \mathbf{R}}{\partial v} = -a\cosh u \sin v\mathbf{i}_1 + a\sinh u \cos v\mathbf{i}_2,$$

and $\frac{\partial \mathbf{R}}{\partial z} = \mathbf{i}_3$. We see that the three vectors are orthogonal pairwise.

(d) Show that the arc length in the new coordinate system is given by

$$ds^2 = a^2(\cosh^2 u - \cos^2 v)(du^2 + dv^2) + dz^2.$$

Show. All $g_{ij} = 0$ for $i \neq j$. We find

$$g_{11} = \frac{\partial \mathbf{R}}{\partial u} \cdot \frac{\partial \mathbf{R}}{\partial u} = (a\sinh u \cos v)^2 + (a\cosh u \sin v)^2 = a^2(\cosh^2 u - \cos^2 v).$$

Similarly we find $g_{22} = g_{11}$ and $g_{33} = 1$.

3. Consider the new coordinates u, v, w defined by

$$u = x - y, \quad v = y + z, \quad w = x - z$$

(a) Find the inverse transformation. (Solution: Use Gause elimination to find $x = (u + v + w)/2,\ y = (-u + v + w)/2,\ z = (u + v - w)/2.$)

(b) Show that the coordinate curves are straight lines.

(Solution: Use linear equation concept)

(c) Show that the coordinate system (u, v, w) is not orthogonal. (Combining (b) and (c), we call this an *oblique* coordinate system.)

(Solution: Show $g_{12} \neq 0$ for example.)

(d) Show that the u, v, w coordinate axes are left-handed.

(Solution: $\partial \mathbf{R}/\partial u \cdot (\partial \mathbf{R}/\partial v \times \partial \mathbf{R}/\partial w) = -1/2 < 0$.)

(e) Find the expression ds of the arc length in the coordinates (u, v, w).

Solution. $ds^2 = 3(du^2 + dv^2 + dw^2)/4 + (du\,dv + dv\,dw - dw\,du)/2$.

4. Find the expression of ∇f for $f = xy + z$ in cylindrical coordinate system.

Solution. In cylindrical coordinates: $u_1 = r$, $u_2 = \theta$, $u_3 = z$, the function becomes $f = r^2 \sin\theta \cos\theta + z$. The formula is

$$\nabla = \mathbf{u}_1 \frac{\partial}{\partial r} + \mathbf{u}_2 \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{u}_3 \frac{\partial}{\partial z}$$

where

$$\mathbf{u}_1 = \cos\theta \mathbf{i}_1 + \sin\theta \mathbf{i}_2, \ \mathbf{u}_2 = -\sin\theta \mathbf{i}_1 + \cos\theta \mathbf{i}_2, \ \mathbf{u}_2 = \mathbf{i}_3.$$

Applying ∇ to the given scalar field f, we find

$$\nabla f = r \sin(2\theta) \mathbf{u}_1 + r(\cos^2\theta - \sin^2\theta)\mathbf{u}_2 + \mathbf{u}_3.$$

5. Find div $\mathbf{F}$ in spherical coordinates where

$$\mathbf{F} = r\mathbf{u}_r + \sin\theta \mathbf{u}_\phi + r\cos\theta \mathbf{u}_\theta.$$

Solution. Recall the distance formula from Lecture 13

$$(ds)^2 = (dr)^2 + (rd\phi)^2 + (r\sin\phi d\theta)^2$$

so that $h_1 = 1, h_2 = r, h_3 = r\sin\phi$. Recall the div formula from Lecture 14

$$\text{div } \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(F_1 h_2 h_3) + \frac{\partial}{\partial u_2}(F_2 h_1 h_3) + \frac{\partial}{\partial u_3}(F_3 h_1 h_2)\right].$$

We have $F_1 = r, F_2 = \sin\theta, F_3 = r\cos\theta$. Let $u_1 = r, u_2 = \phi, u_3 = \theta$. Then

$$\frac{\partial}{\partial u_1}(F_1 h_2 h_3) = 3r^2 \sin\phi,\ \frac{\partial}{\partial u_2}(F_2 h_1 h_3) = r\sin\theta\cos\phi,\ \frac{\partial}{\partial u_3}(F_3 h_1 h_2) = -r^2\sin\theta.$$

Thus

$$\text{div } \mathbf{F} = \frac{1}{r\sin\phi}[3r\sin\phi + \sin\theta\cos\phi - r\sin\theta].$$

6. (Optional problem) Find the expression of $\nabla^2 f$ in spherical coordinates where $f(x, y, z) = xy + yz + zx$.

Solution. The spherical coordinates are

$$\begin{aligned} x &= r\sin\phi\cos\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\phi. \end{aligned}$$

In spherical coordinates, the function $f(x, y, z) = xy + yz + zx$ is

$$f = \frac{1}{2}r^2[\sin(2\theta)\sin^2\phi + \sin\theta\sin(2\phi) + \cos\theta\sin(2\phi)].$$

We already know that $h_1 = 1, h_2 = r, h_3 = r\sin\phi$. From Lecture 14 notes we know that $\nabla^2 f = \nabla\cdot\nabla f = \text{div } \nabla f = \Delta f$ has formula

$$\Delta f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right].$$

We calculate that

$$\frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right) = 3[\sin(2\theta)\sin^2\phi + \sin\theta\sin(2\phi) + \cos\theta\sin(2\phi)].$$

$$\begin{aligned} \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial u_2}\right) &= \frac{1}{2\sin\phi}\{\cos\phi[\sin(2\theta)\sin(2\phi) + 2\sin\theta\cos(2\phi) + 2\cos\theta\cos(2\phi)] \\ &\quad + \sin\phi[2\sin(2\theta)\cos(2\phi) - 4\sin\theta\sin(2\phi) - 4\cos\theta\sin(2\phi)]\}. \end{aligned}$$

$$\frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right) = -\frac{1}{\sin\phi}[2\sin(2\theta)\sin\phi + \sin\theta\cos\phi + \cos\theta\cos\phi].$$

Adding the three terms yields the answer to our question.