M597K: Homework Assignment 5

Date: September 25, Wednesday, 2002. Due Friday, October 4.

1. Recall that the product of two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ is defined as the matrix $AB = (c_{ij})$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad (i, j = 1, 2, \cdots, n).$$

Thus show that

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \frac{\partial x_1}{\partial u_3} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_2}{\partial u_3} \\ \frac{\partial x_3}{\partial u_1} & \frac{\partial x_3}{\partial u_2} & \frac{\partial x_3}{\partial u_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (1)$$

Here (x_1, x_2, x_3) represents cartesian coordinates and (u_1, u_2, u_3) represents curvilinear coordinates whose Jacobian is not zero. (From this equation, and the rule

$$\det\,(AB) = \det\,(A)\,\det(B),$$

one can easily deduce that the Jacobian of the inverse transformation is the reciprocal of the Jacobian of the (forward) transformation: i.e., identity (4) in Section 1.15, Lecture 12.)

2. The transformation relating the cartesian coordinates x, y, z to the elliptic cylindrical coordinates u, v, z is given by the equations

$$x = a\cosh u \cos v, \quad y = a \sinh u \sin v, \quad z = z$$

$(u \geq 0, 0 \leq v < 2\pi, a > 0$ constant $)$.

(a) Show that in the xy-plane a curve $u =$ constant represents an ellipse, while a curve $v =$ constant represents half of one branch of a hyperbola.

(b) Sketch each curve on the xy-plane corresponding to the values $u = 0$; $v = 0$; $v = \pi$; $v = \pi/2$; respectively.

(c) Verify that the new coordinate system is orthogonal.

(d) Show that the arc length in the new coordinate system is given by

$$ds^2 = a^2(\cosh^2 u - \cos^2 v)(du^2 + dv^2) + dz^2.$$

3. Consider the new coordinates u, v, w defined by

$$u = x - y, \quad v = y + z, \quad w = x - z$$

(a) Find the inverse transformation.

(b) Show that the coordinate curves are straight lines.

(c) Show that the coordinate system (u, v, w) is not orthogonal. (Combining (b) and (c), we call this an *oblique* coordinate system.)

(d) Show that the u, v, w coordinate axes are left-handed.

(e) Find the expression ds of the arc length in the coordinates (u, v, w).

4. Find the expression of ∇f for $f = xy + z$ in cylindrical coordinate system.

5. Find div $\mathbf{F}$ in spherical coordinates where

$$\mathbf{F} = r\mathbf{u}_r + \sin\theta \mathbf{u}_\phi + r\cos\theta \mathbf{u}_\theta.$$

6. (Optional problem) Find the expression of $\nabla^2 f$ in spherical coordinates where $f(x, y, z) = xy + yz + zx$.