M597K: Solution to Homework 4

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1. Given a scalar $\Phi(x_1, x_2, x_3)$, does the gradient $\nabla\Phi = (\partial_{x_1}\Phi, \partial_{x_2}\Phi, \partial_{x_3}\Phi)$ satisfy the law of coordinate transformation for first-order tensors? That is,

$$\partial_{x'_i}\Phi'(x'_1, x'_2, x'_3) = \alpha_{i'k}\partial_{x_k}\Phi(x_1, x_2, x_3)?$$

Here $\alpha_{i'k}$ is the coordinate transformation from one rectangular coordinate system K to another rectangular coordinate system K'. Show your work. But you can skip this homework if you know how to solve the next problem.

Solution. (10 points each) Yes, it does. We have the inverse transformation

$$x_k = \alpha_{j'k}x'_j + x'_{0k}.$$

And the chain rule:

$$\frac{\partial}{\partial x'_i} = \frac{\partial x_k}{\partial x'_i}\frac{\partial}{\partial x_k} = \alpha_{i'k}\frac{\partial}{\partial x_k}.$$

Thus from $\Phi'(x'_1, x'_2, x'_3) = \Phi(x_1, x_2, x_3)$ and differentiation, we have

$$\begin{aligned} \partial_{x'_i}\Phi'(x'_1, x'_2, x'_3) &= \partial_{x'_i}\Phi(x_1, x_2, x_3) \\ &= \alpha_{i'k}\tfrac{\partial}{\partial x_k}\Phi(x_1, x_2, x_3). \end{aligned} \tag{1}$$

That is the law for transformation of first-order tensors.

2. Given a scalar function $\Phi = \Phi(x_1, x_2, x_3)$, do the quantities

$$\frac{\partial^2\Phi}{\partial x_i \partial x_k}$$

form a tensor? Show your work.

Solution. Yes, it does. Continuing our solution in problem 1, we have

$$\begin{aligned} \partial_{x'_j}\partial_{x'_i}\Phi'(x'_1, x'_2, x'_3) &= \partial_{x'_j}\partial_{x'_i}\Phi(x_1, x_2, x_3) \\ &= \partial_{x'_j}[\alpha_{i'k}\tfrac{\partial}{\partial x_k}\Phi(x_1, x_2, x_3)] \\ &= \alpha_{j'm}\tfrac{\partial}{\partial x_m}[\alpha_{i'k}\tfrac{\partial}{\partial x_k}\Phi(x_1, x_2, x_3)] \\ &= \alpha_{j'm}\alpha_{i'k}\tfrac{\partial^2\Phi(x_1, x_2, x_3)}{\partial x_m \partial x_k} \end{aligned} \tag{2}$$

which is the law for transformation of second-order tensors.

3. The stress tensor at a point has components given by

$$(p_{ij}) \quad = \quad \begin{pmatrix} 1 & -2 & 2 \\ -2 & 3 & 0 \\ 2 & 0 & -1 \end{pmatrix}.$$

Find the stress vector ($\mathbf{p}_b$) across an area normal to the unit vector

$$\mathbf{b} = (\mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3)/\sqrt{3}.$$

What is the normal stress across such an area (i.e, the projection $(\mathbf{p}_b \cdot \mathbf{b})\mathbf{b}$ of the vector $\mathbf{p}_b$ on to $\mathbf{b}$)?

Solution. A formula for $\mathbf{p}_b$ is $\mathbf{p}_b = \mathbf{p}_i b_i$ where $\mathbf{p}_i$ are the rows (or columns) of the matrix of the stress tensor. Thus

$$\mathbf{p}_b = (1, -2, 2)/\sqrt{3} + (-2, 3, 0)(-1)/\sqrt{3} + (2, 0, -1)/\sqrt{3} = (5, -5, 1)/\sqrt{3}.$$

The normal stress across such an area is

$$(\mathbf{p}_b \cdot \mathbf{b})\mathbf{b} = (11/3)\mathbf{b} = \frac{11\sqrt{3}}{9}(1, -1, 1).$$

4. For the stress tensor given in the previous problem,

(a) What is the total force on a unit disk whose normal is in the positive x_2 direction? *Answer*: the second column (or row) of the matrix (p_{ij}).

(b) What is the x_3 component of the total force on a unit disk whose normal is in the positive x_1 direction? *Answer* : Either p_{13} or p_{31}, which are equal.

5. The unit base vectors $\mathbf{i}'_i$ of a new coordinate system K' are given by

$$\mathbf{i}'_1 = \frac{\mathbf{i}_2 + \mathbf{i}_3}{\sqrt{2}}, \ \mathbf{i}'_2 = \frac{\mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3}{\sqrt{3}}, \ \mathbf{i}'_3 = \frac{2\mathbf{i}_1 + \mathbf{i}_2 - \mathbf{i}_3}{\sqrt{6}}.$$

The stress tensor p_{ik} in the system K is of the form

$$(p_{ik}) \quad = \quad \begin{pmatrix} p_1 & 0 & p_2 \\ 0 & 0 & 0 \\ p_2 & 0 & p_3 \end{pmatrix}.$$

Find the component p'_{13} of the stress tensor p'_{lm} in K'.

Solution. The $\alpha_{i'j} = \mathbf{i}'_i \cdot \mathbf{i}_j$ are found to be given by

$$(\alpha_{i'j}) = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \end{pmatrix}.$$

By formula of transformation, we have

$$\begin{aligned} p'_{13} &= \alpha_{1'k}\alpha_{3'm}p_{km} \\ &= \alpha_{1'1}(\alpha_{3'1}p_{11} + \alpha_{3'2}p_{12} + \alpha_{3'3}p_{13}) + \\ &\quad \alpha_{1'2}(\alpha_{3'1}p_{21} + \alpha_{3'2}p_{22} + \alpha_{3'3}p_{23}) + \\ &\quad \alpha_{1'3}(\alpha_{3'1}p_{31} + \alpha_{3'2}p_{32} + \alpha_{3'3}p_{33}) \\ &= 0(\cdots) + \\ &\quad 1/\sqrt{2}(\alpha_{3'1}0 + \alpha_{3'2}0 + \alpha_{3'3}0) + \\ &\quad 1/\sqrt{2}(\alpha_{3'1}p_2 + \alpha_{3'2}0 + \alpha_{3'3}p_3) \\ &= 1/\sqrt{2}(\alpha_{3'1}p_2 + \alpha_{3'3}p_3) \\ &= 1/(2\sqrt{3})(2p_2 - p_3) = (2p_2 - p_3)/(2\sqrt{3}). \end{aligned} \tag{3}$$

Using the matrix multiplication $(p'_{ij}) = (\alpha)(p)(\alpha)^T$, one can find all components together. Here $(\alpha)^T$ denotes the transpose of (α).

6. Let a_i, b_j, and c_k be the components of three vectors. Verify that the 27 quantities

$$d_{ijk} = 2a_i b_j c_k$$

form a tensor of order 3.

Solution. We know $a'_i = \alpha_{i'l}a_l, b'_j = \alpha_{j'm}b_m, c'_k = \alpha_{k'n}c_n$, thus

$$d'_{ijk} = 2a'_i b'_j c'_k = 2\alpha_{i'l}a_l\alpha_{j'm}b_m\alpha_{k'n}c_n = \alpha_{i'l}\alpha_{j'm}\alpha_{k'n}(2a_l b_m c_n) = \alpha_{i'l}\alpha_{j'm}\alpha_{k'n}d_{lmn}.$$

7. Form a scalar by contracting the tensor with the matrix

$$\begin{pmatrix} -5 & 0 & 1 \\ -1 & 3 & 7 \\ 4 & 8 & 2 \end{pmatrix}. \quad \text{Answer} : -5 + 3 + 2 = 0.$$

8. Given that

$$(T_{ik}) \quad = \quad \begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & 3 \\ -4 & 0 & 2 \end{pmatrix}, \qquad \mathbf{A} = \mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3, \qquad \mathbf{B} = 2\mathbf{i}_1 + \mathbf{i}_2 - \mathbf{i}_3.$$

Find the inner products $T_{ik}A_i$, $T_{ik}A_k$, and $T_{ik}A_iB_k$. If it is too tedious for you, you may choose i or k to be 2 if that i or k is not a dummy variable. (Note: First index in T is the row number.)

Solution. $T_{ik}A_i = T_{1k}A_1 + T_{2k}A_2 + T_{3k}A_3$.
For $k = 1$, we have $T_{i1}A_i = 2(1) + 0(-1) + (-4)(1) = -2$.
For $k = 2$, we have $T_{i2}A_i = (-1)1 + 1(-1) + 0(1) = -2$.
For $k = 3$, we have $T_{i3}A_i = 2(1) + 3(-1) + 2(1) = 1$.
For $i = 1$, $T_{ik}A_k = (2, -1, 2) \cdot (1, -1, 1) = 5$.
For $i = 2$, $T_{ik}A_k = (0, 1, 3) \cdot (1, -1, 1) = 2$.
For $i = 3$, $T_{ik}A_k = (-4, 0, 2) \cdot (1, -1, 1) = -2$.
You can use matrix multiplication to simplify the writing.

9. Show that the delta function $\delta_{ij} = \mathbf{i}_i \cdot \mathbf{i}_j$ satisfies the law of transformation for second-order tensors, where $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are the unit vectors of a rectangular coordinate system K. This delta tensor is called the *unit tensor*.

Proof. Recall $\mathbf{i}'_k = \alpha_{k'm}\mathbf{i}_m$.Thus

$$\delta'_{ij} = \mathbf{i}'_i \cdot \mathbf{i}'_j = (\alpha_{i'm}\mathbf{i}_m) \cdot (\alpha_{j'n}\mathbf{i}_n) = \alpha_{i'm}\alpha_{j'n}\mathbf{i}_m \cdot \mathbf{i}_n = \alpha_{i'm}\alpha_{j'n}\delta_{mn}.$$

So the delta satisfies the law of transformation for second-order tensors.

10. (Optional) In nonlinear elasticity, the strain tensor is more accurately defined to be

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right)$$

where u_i are again components of the displacement vector. Show it still satisfies the transformation law for second-order tensors.

(This set of homework looks a lot, but try your best. Some of them are really easy.)

Proof. Recall $u'_i = \alpha_{i'k} u_k$, and

$$u'_{ik} = \frac{1}{2}(\frac{\partial u'_i}{\partial x'_k} + \frac{\partial u'_k}{\partial x'_i} + \frac{\partial u'_l}{\partial x'_i}\frac{\partial u'_l}{\partial x'_k}),$$

and

$$\frac{\partial}{\partial x'_i} = \frac{\partial x_j}{\partial x'_i}\frac{\partial}{\partial x_j} = \alpha_{i'j}\frac{\partial}{\partial x_j}.$$

Thus

$$\begin{aligned} 2u'_{ik} = & \tfrac{\partial u'_i}{\partial x'_k} + \tfrac{\partial u'_k}{\partial x'_i} + \tfrac{\partial u'_l}{\partial x'_i}\tfrac{\partial u'_l}{\partial x'_k} \\ = & \alpha_{k'm}\tfrac{\partial}{\partial x_m}(\alpha_{i'n}u_n) + \alpha_{i'n}\tfrac{\partial}{\partial x_n}(\alpha_{k'm}u_m) + \\ & +\alpha_{i'n}\tfrac{\partial}{\partial x_n}(\alpha_{l's}u_s)\alpha_{k'm}\tfrac{\partial}{\partial x_m}(\alpha_{l'r}u_r) \\ = & \alpha_{k'm}\alpha_{i'n}\tfrac{\partial u_n}{\partial x_m} + \alpha_{i'n}\alpha_{k'm}\tfrac{\partial u_m}{\partial x_n} + \\ & \alpha_{i'n}\alpha_{k'm}\alpha_{l's}\alpha_{l'r}\tfrac{\partial u_s}{\partial x_n}\tfrac{\partial u_r}{\partial x_m} \\ = & \alpha_{k'm}\alpha_{i'n}(\tfrac{\partial u_n}{\partial x_m} + \tfrac{\partial u_m}{\partial x_n} + \delta_{rs}\tfrac{\partial u_s}{\partial x_n}\tfrac{\partial u_r}{\partial x_m}) \\ = & \alpha_{k'm}\alpha_{i'n}(\tfrac{\partial u_n}{\partial x_m} + \tfrac{\partial u_m}{\partial x_n} + \tfrac{\partial u_r}{\partial x_n}\tfrac{\partial u_r}{\partial x_m}) \\ = & \alpha_{i'n}\alpha_{k'm}(2u_{nm}), \end{aligned} \tag{4}$$

where we've used the orthogonality condition $\alpha_{l's}\alpha_{l'r} = \delta_{rs}$. That is the transformation law for second-order tensors.