M597K: Homework Assignment 4

Date: Sept. 18, Wed., 2002. Due Friday Sept 27.

1. Given a scalar $\Phi(x_1, x_2, x_3)$, does the gradient $\nabla\Phi = (\partial_{x_1}\Phi, \partial_{x_2}\Phi, \partial_{x_3}\Phi)$ satisfy the law of coordinate transformation for first-order tensors? That is,

$$\partial_{x'_i}\Phi'(x'_1, x'_2, x'_3) = \alpha_{i'k}\partial_{x_k}\Phi(x_1, x_2, x_3)?$$

Here $\alpha_{i'k}$ is the coordinate transformation from one rectangular coordinate system K to another rectangular coordinate system K'. Show your work. But you can skip this homework if you know how to solve the next problem.

2. Given a scalar function $\Phi = \Phi(x_1, x_2, x_3)$, do the quantities

$$\frac{\partial^2\Phi}{\partial x_i \partial x_k}$$

form a tensor? Show your work.

3. The stress tensor at a point has components given by

$$(p_{ij}) \quad = \quad \begin{pmatrix} 1 & -2 & 2 \\ -2 & 3 & 0 \\ 2 & 0 & -1 \end{pmatrix}.$$

Find the stress vector $(\mathbf{p}_b)$ across an area normal to the unit vector

$$\mathbf{b} = (\mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3)/\sqrt{3}.$$

What is the normal stress across such an area (i.e, the projection $(\mathbf{p}_b \cdot \mathbf{b})\mathbf{b}$ of the vector $\mathbf{p}_b$ on to $\mathbf{b}$)?

4. For the stress tensor given in the previous problem,

(a) What is the total force on a unit disk whose normal is in the positive x_2 direction?

(b) What is the x_3 component of the total force on a unit disk whose normal is in the positive x_1 direction?

5. The unit base vectors $\mathbf{i}'_i$ of a new coordinate system K' are given by

$$\mathbf{i}'_1 = \frac{\mathbf{i}_2 + \mathbf{i}_3}{\sqrt{2}},\ \mathbf{i}'_2 = \frac{\mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3}{\sqrt{3}},\ \mathbf{i}'_3 = \frac{2\mathbf{i}_1 + \mathbf{i}_2 - \mathbf{i}_3}{\sqrt{6}}.$$

The stress tensor p_{ik} in the system K is of the form

$$(p_{ik}) \quad = \quad \begin{pmatrix} p_1 & 0 & p_2 \\ 0 & 0 & 0 \\ p_2 & 0 & p_3 \end{pmatrix}.$$

Find the component p'_{13} of the stress tensor p'_{lm} in K'.

6. Let a_i, b_j, and c_k be the components of three vectors. Verify that the 27 quantities $d_{ijk} = 2a_i b_j c_k$ form a tensor of order 3.

7. Form a scalar by contracting the tensor which is given by the matrix

$$\begin{pmatrix} -5 & 0 & 1 \\ -1 & 3 & 7 \\ 4 & 8 & 2 \end{pmatrix}.$$

8. Given that

$$(T_{ik}) \quad = \quad \begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & 3 \\ -4 & 0 & 2 \end{pmatrix}, \qquad \mathbf{A} = \mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3, \qquad \mathbf{B} = 2\mathbf{i}_1 + \mathbf{i}_2 - \mathbf{i}_3.$$

Find the inner products $T_{ik}A_i$, $T_{ik}A_k$, and $T_{ik}A_iB_k$. If it is too tedious for you, you may choose i or k to be 2 if that i or k is not a dummy variable. (Note: First index in T is the row number.)

9. Show that the delta function $\delta_{ij} = \mathbf{i}_i \cdot \mathbf{i}_j$ satisfies the law of transformation for second-order tensors, where $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are the unit vectors of a rectangular coordinate system K. This delta tensor is called the *unit tensor*.

10. (Optional) In nonlinear elasticity, the strain is more accurately defined to be

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right)$$

where u_i are again components of the displacement vector. Show it still satisfies the transformation law for second-order tensors.

(This set of homework looks a lot, but try your best. Some of them are really easy.)