M597K: Solution to Homework 3

Date: Sept. 20, Friday

Solutions 1-2: Omitted.

3. (Summation convention) Expand the terms $A_i B^k C_i$ and $a_{ij} b_j$. Is there a summation in $a_i + b_i$?

Solution: (20 points) $A_i B^k C_i = B^k(A_1 C_1 + A_2 C_2 + A_3 C_3)$.

$a_{ij} b_j = a_{i1} b_1 + a_{i2} b_2 + a_{i3} b_3$.

There is no summation in $a_i + b_i$ since the repeated index i is not in a product.

4. The new coordinate system K' is obtained by rotating the $\mathbf{i}_i$ coordinate system an angle θ about the x_3 axis counterclockwise. Find the coefficients (i.e., $\alpha_{i'j}$ and $\alpha_{j'i}$) in the equations:

$$x'_i = \alpha_{i'j} x_j, \qquad\qquad x_i = \alpha_{j'i} x'_j.$$

Solution: (20 points) Once you draw the three-dimensional graphs of the coordinate unit vectors, you can find

$$\begin{aligned} \mathbf{i}'_1 &= \cos\theta \mathbf{i}_1 + \sin\theta \mathbf{i}_2, \\ \mathbf{i}'_2 &= -\sin\theta \mathbf{i}_1 + \cos\theta \mathbf{i}_2, \\ \mathbf{i}'_3 &= \mathbf{i}_3. \end{aligned}$$

Then by definition of the $\alpha_{i'j}$ and $\alpha_{j'i}$, one can find

$$x'_1 = \cos\theta x_1 + \sin\theta x_2; \quad x'_2 = -\sin\theta x_1 + \cos\theta x_2; \quad x'_3 = x_3.$$

And by rotating backward, one finds

$$x_1 = \cos\theta x'_1 - \sin\theta x'_2; \quad x_2 = \sin\theta x'_1 + \cos\theta x'_2; \quad x_3 = x'_3.$$

5. The unit base vectors $\mathbf{i}'_i$ of a new coordinate system K' are given by

$$\mathbf{i}'_1 = \frac{\mathbf{i}_2}{\sqrt{3}} + \frac{2\mathbf{i}_3}{\sqrt{6}}, \ \mathbf{i}'_2 = \frac{\mathbf{i}_1}{\sqrt{2}} - \frac{\mathbf{i}_2}{\sqrt{3}} + \frac{\mathbf{i}_3}{\sqrt{6}}, \ \mathbf{i}'_3 = \frac{\mathbf{i}_1}{\sqrt{2}} + \frac{\mathbf{i}_2}{\sqrt{3}} - \frac{\mathbf{i}_3}{\sqrt{6}}.$$

Find the coefficients in the equation: $x'_i = \alpha_{i'j} x_j$.

Solution: (20 points) One can use the formula $\alpha_{i'j} = \mathbf{i}'_i \cdot \mathbf{i}_j$ or use the similarity between the expressions of $\mathbf{i}'_i$ and x'_i to draw conclusion

$$(\alpha_{i'j}) = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix}.$$

6. Given the transformation of coordinates

$$x'_i = \alpha_{i'j}x_j$$

where

$$(\alpha_{i'j}) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{6} \end{pmatrix}.$$

If a vector $\mathbf{v}$ has the components $(2, -3, 6)$ with respect to the x_i-coordinate system, find its components in the x'_i-system.

Solution: (20 points) One can use the formula $x'_i = \alpha_{i'j}x_j$ or the matrix notation $(\alpha_{i'j})\mathbf{v}^T$ to find $x' = (4\sqrt{2}, \frac{\sqrt{3}}{3}, -\frac{5\sqrt{6}}{3})$.

7. Given the second-order tensor

$$(a_{ij}) = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & 2 \\ -3 & -2 & 0 \end{pmatrix}.$$

Find the components a'_{21} and a'_{33} of this tensor in the coordinate system x'_i defined by $x'_i = \alpha_{i'j}x_j$ where

$$(\alpha_{i'j}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If you can use computer, then find all the components a'_{ij}. (The formula in matrix notation is $(A') = (\alpha)(A)(\alpha)^T$.)

Solution: (20 points) One can use $a'_{ij} = \alpha_{i'k}\alpha_{j'm}a_{km}$ or the matrix way to find

$$(a'_{ij}) = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}.$$

So $a'_{21} = 1$ and $a'_{33} = 0$.