M597K: Homework Assignment 3

Date: Sept. 11, Wed., 2002. Due Friday Sept 20.

1. A coordinate-independent representation of the gradient is given by

$$\nabla\phi(\mathbf{x}) = \lim_{V\to 0}\frac{1}{V}\iint_{\partial V}\mathbf{n}(\mathbf{y})\phi(\mathbf{y})\,dS_{\mathbf{y}}$$

where V is a domain that contains the point $\mathbf{x}$ and $\mathbf{n}$ is the unit exterior normal to ∂V. Either give a proof of this formula or say "Yes, I have read the proof from the text, and understand it."

2. Consider a rigid body rotating about a fixed point O with angular velocity $\mathbf{w}$. See Figure 1.6.3. The velocity of a point with position vector $\mathbf{r}$ is given by

$$\mathbf{v} = \mathbf{w}\times\mathbf{r}.$$

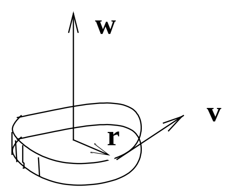

Figure 1.6.3. Curl is twice angular velocity.

Show that

$$\begin{aligned}\text{curl}_2\mathbf{v} &= 2w_2 \\ \text{curl}_3\mathbf{v} &= 2w_3.\end{aligned} \tag{1}$$

Combined with the calculation for $\text{curl}_1\mathbf{v} = 2w_1$ done in Lecture 6, one can conclude that

$$\text{curl }\mathbf{v} = 2w.$$

3. (Summation convention) Expand the terms $A_iB^kC_i$ and $a_{ij}b_j$. Is there a summation in $a_i + b_i$?

4. The new coordinate system K' is obtained by rotating the $\mathbf{i}_i$ coordinate system an angle θ about the x_3 axis counterclockwise. Find the coefficients (i.e., $\alpha_{i'j}$ and $\alpha_{j'i}$) in the equations:

$$\begin{aligned}x'_i &= \alpha_{i'j}x_j \\ x_i &= \alpha_{j'i}x'_j.\end{aligned}$$

5. The unit base vectors $\mathbf{i}'_i$ of a new coordinate system K' are given by

$$\mathbf{i}'_1 = \frac{\mathbf{i}_2}{\sqrt{3}} + \frac{2\mathbf{i}_3}{\sqrt{6}},\ \mathbf{i}'_2 = \frac{\mathbf{i}_1}{\sqrt{2}} - \frac{\mathbf{i}_2}{\sqrt{3}} + \frac{\mathbf{i}_3}{\sqrt{6}},\ \mathbf{i}'_3 = \frac{\mathbf{i}_1}{\sqrt{2}} + \frac{\mathbf{i}_2}{\sqrt{3}} - \frac{\mathbf{i}_3}{\sqrt{6}}.$$

Find the coefficients in the equation: $x'_i = \alpha_{i'j} x_j$.

6. Given the transformation of coordinates

$$x'_i = \alpha_{i'j} x_j$$

where

$$(\alpha_{i'j}) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{6} \end{pmatrix}.$$

If a vector $\mathbf{v}$ has the components $(2, -3, 6)$ with respect to the x_i-coordinate system, find its components in the x'_i-system.

7. Given the second-order tensor

$$(a_{ij}) = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & 2 \\ -3 & -2 & 0 \end{pmatrix}.$$

Find the components a'_{21} and a'_{33} of this tensor in the coordinate system x'_i defined by $x'_i = \alpha_{i'j} x_j$ where

$$(\alpha_{i'j}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If you can use computer, then find all the components a'_{ij}. (The formula in matrix notation is $(A') = (\alpha)(A)(\alpha)^T$.)