

M597K: Solution to Homework Assignment 2

Date: Friday, Sept. 13, 2002.

1. Find the derivative of the vector $\mathbf{A}(t) = (\cos t, \sin t, 2t)$. Draw the graph of $\mathbf{A}(t)$ with $\mathbf{A}'(t)$.

Solution. (10 points) The derivative is $\mathbf{A}'(t) = (-\sin t, \cos t, 2)$. The graph is in Figure 1.

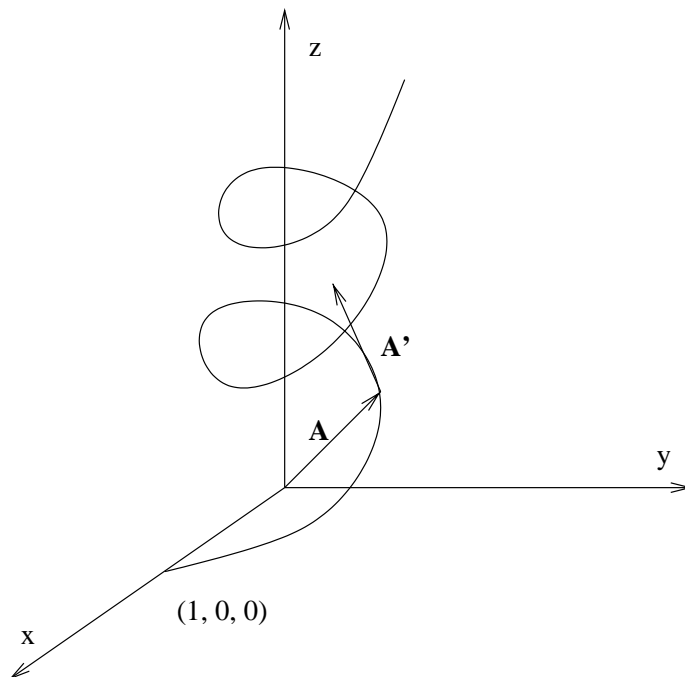


Figure 1: Solution to homework 2, problem 1.

2. Integrate the vector $\mathbf{B}(t) = (e^t, \sin t, 2t)$ to find $\int_0^1 \mathbf{B}(t) dt$.

Solution. (10 points)

$$\int_0^1 \mathbf{B}(t) dt = \left(\int_0^1 e^t dt, \int_0^1 \sin t dt, \int_0^1 2t dt \right) = (e - 1, 1 - \cos 1, 1) \approx (1.718, 0.46, 1).$$

3. Evaluate the line integral

$$\int_L \mathbf{C} \cdot d\mathbf{r}$$

where $\mathbf{C} = (x_2, -x_1, -1)$ and L is a directed curve given by the graph of the vector $\mathbf{A}(t)$ in Exercise 1 from $t = 0$ to $t = 2\pi$.

Solution. (10 points) We need to use the parametrization:

$$d\mathbf{r} = (-\sin t, \cos t, 2) dt, \quad \mathbf{C} = (x_2, -x_1, -1) = (\sin t, -\cos t, -1).$$

Then,

$$\int_L \mathbf{C} \cdot d\mathbf{r} = \int_0^{2\pi} (-\sin^2 t - \cos^2 t - 2) dt = \int_0^{2\pi} -3 dt = -6\pi.$$

4. Find the total circulation

$$\oint_C (x_1 + x_2)dx_1 + (x_1 - x_2)dx_2$$

where C is the ellipse $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$ going counter-clockwise.

Solution. (10 points) Method I: Let $x_1 = a \cos t$, $x_2 = b \sin t$. Then

$$\begin{aligned} & \oint_C (x_1 + x_2) dx_1 + (x_1 - x_2) dx_2 \\ &= \int_0^{2\pi} -(a \cos t + b \sin t)a \sin t dt + (a \cos t - b \sin t)b \cos t dt \\ &= \int_0^{2\pi} [ab(\cos^2 t - \sin^2 t) - (a^2 + b^2) \cos t \sin t] dt \\ &= ab \int_0^{2\pi} \cos(2t) dt + \frac{a^2 + b^2}{2} \int_0^{2\pi} \sin(2t) dt = 0. \end{aligned} \tag{1}$$

Method II: Let $P = x_1 + x_2$, $Q = x_1 - x_2$. Then, by green's formula, one has

$$\begin{aligned} & \oint_C (x_1 + x_2)dx_1 + (x_1 - x_2)dx_2 \\ &= \oint_C Pdx_1 + Qdx_2 \\ &= \int \int_S \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2} \right) dS = \int \int_S (1 - 1) dS = 0. \end{aligned} \tag{2}$$

5. Find the gradient of $f = x_1^2 + x_2^2 + x_3^2$.

Solution. (10 points) $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right) = (2x_1, 2x_2, 2x_3)$.

6. Find a unit normal vector to the surface

$$x_3 = 2 - x_1 - x_2^2.$$

Solution. (10 points) Let $\phi = x_1 + x_2^2 + x_3$. Then Our surface is the level surface $\phi = 2$. We know that the gradient of ϕ is a vector normal to the level surfaces. So a normal vector to the surface $x_3 = 2 - x_1 - x_2^2$ is

$$\nabla \phi = (1, 2x_2, 1).$$

A unit normal vector is then

$$(1, 2x_2, 1) / \sqrt{2 + 4x_2^2}.$$

If your answer is the negative of the answer here, it is also right.

7. Find the divergence and curl of the vector field

$$\mathbf{A} = (x_2x_3, x_1x_3, x_1x_2).$$

Solution. (10 points)

$$\begin{aligned} \operatorname{div} \mathbf{A} &= \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} \\ &= \frac{\partial(x_2x_3)}{\partial x_1} + \frac{\partial(x_1x_3)}{\partial x_2} + \frac{\partial(x_1x_2)}{\partial x_3} = 0. \end{aligned} \quad (3)$$

$$\operatorname{curl} \mathbf{A} = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ x_2x_3 & x_1x_3 & x_1x_2 \end{vmatrix} = \mathbf{0}.$$

8. Find the total flux of the vector field $\mathbf{A} = (x_1, x_2, x_3)$ out of the unit sphere: $x_1^2 + x_2^2 + x_3^2 = 1$.

Solution. (10 points) Method I: Recall that for the sphere $x^2 + y^2 + z^2 = r^2$ a unit normal vector is $\mathbf{n} = \mathbf{r}/|\mathbf{r}|$ where $\mathbf{r} = (x, y, z)$. For this problem $\mathbf{n} = (x_1, x_2, x_3) = \mathbf{A}$, so $\mathbf{A} \cdot \mathbf{n} = 1$. Thus,

$$\text{flux} = \int \int_S \mathbf{A} \cdot \mathbf{n} dS = \int \int_S 1 dS = \text{Area}(S) = 4\pi.$$

Method II: $\operatorname{div} \mathbf{A} = 1 + 1 + 1 = 3$. Since \mathbf{A} is not singular in the entire unit ball, we can use Gauss formula:

$$\text{flux} = \int \int_S \mathbf{A} \cdot \mathbf{n} dS = \int \int \int_V \operatorname{div} \mathbf{A} dV = 3 \int \int \int_V 1 dV = 4\pi.$$

9. Evaluate the line integral by using Green's theorem:

$$\oint_C (x^2 + y^2)dx + 2xydy,$$

where C is the square bounded by the lines $x = 0, x = 2, y = 0, y = 2$.

Solution. (10 points) Let $P = x^2 + y^2, Q = 2xy$. Then

$$\begin{aligned} &\oint_C (x^2 + y^2) dx + 2xy dy \\ &= \oint_C P dx + Q dy = \int \int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dS \\ &= \int \int_S (2y - 2y) dS = 0. \end{aligned} \quad (4)$$

10. Let

$$\mathbf{F}(x_1, x_2, x_3) = (x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3)/r^3$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$. Show that the flux of this vector through any closed surface S is 0 if the origin is not enclosed by S .

Solution. (10 points) $\mathbf{F} = (F_1, F_2, F_3)$ where $F_i = x_i/r^3$ $i = 1, 2, 3$.

$$\frac{\partial F_i}{\partial x_i} = \frac{1}{r^6}(r^3 - 3x_i r^2 \frac{\partial r}{\partial x_i}) = \frac{1}{r^4}(r - 3x_i \frac{\partial r}{\partial x_i}) \quad i = 1, 2, 3.$$

$$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} = \frac{3}{r^4}[r - (x_1 \frac{\partial r}{\partial x_1} + x_2 \frac{\partial r}{\partial x_2} + x_3 \frac{\partial r}{\partial x_3})].$$

From $r^2 = x_1^2 + x_2^2 + x_3^2$, we find that $2r \frac{\partial r}{\partial x_k} = 2x_k, k = 1, 2, 3$. Or $\frac{\partial r}{\partial x_k} = x_k/r$. Thus,

$$x_1 \frac{\partial r}{\partial x_1} + x_2 \frac{\partial r}{\partial x_2} + x_3 \frac{\partial r}{\partial x_3} = r \quad \Rightarrow \quad \operatorname{div} \mathbf{F} = 0,$$

except at the origin.

If the origin is not enclosed by S , then \mathbf{F} , and $\operatorname{div} \mathbf{F}$ are not singular in the integration domain. Apply the Gauss formula, we obtain

$$\text{flux} = \int \int_S \mathbf{F} \cdot \mathbf{n} dS = \int \int \int_V \operatorname{div} \mathbf{F} dV = 0.$$