M597K: Solution to Homework Assignment 1

Date: Sept 6, 2002.

1. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a basis (may or may not be orthonormalized). Let $\mathbf{A}$ be a vector. Is it always true that

$$\mathrm{Proj}_{\mathbf{e}_1}\mathbf{A} + \mathrm{Proj}_{\mathbf{e}_2}\mathbf{A} + \mathrm{Proj}_{\mathbf{e}_3}\mathbf{A} = \mathbf{A}?$$

Solution (10 points). No, it is not always true. In particular, it is not true when the basis is not orthonormalized for most vectors $\mathbf{A}$. An example is this. Imagine a cube. And pick a corner, say the left-bottom (front) corner. Name the bottom edge vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, name the vertial edge vector $\mathbf{A}$. Use the long diagonal as $\mathbf{e}_3$; i. e., $\mathbf{e}_3$ is the vector from the bottom-left (front) corner to the top-right (front or back) corner. Now the projection of $\mathbf{A}$ onto either one of $\mathbf{e}_1, \mathbf{e}_2$ is zero. But the projection of $\mathbf{A}$ onto $\mathbf{e}_3$ is not enough to recover $\mathbf{A}$.

2. A parallelogram has acute angle $\pi/4$ and side lengths $a = 2, b = 5$. Thinking of the corresponding sides as vectors $\mathbf{a}$ and $\mathbf{b}$, find

(a) The vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ (what is their geometric meaning?);

(b) The area of the parallelogram.

Solution. (a) (10 points) The diagonal nestled between the sides a and b is the sum $\mathbf{a} + \mathbf{b}$, pointing away from the common point of a and b. The other diagonal, from the tip of $\mathbf{b}$ to the tip of $\mathbf{a}$ is $\mathbf{a} - \mathbf{b}$.

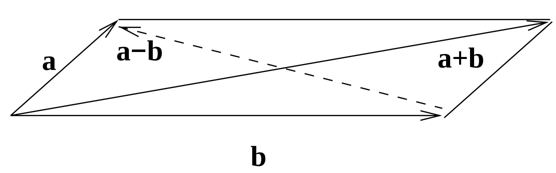

Figure 1. Solution to Homework 1, Problem 2(a).

Solution. (b) (10 points) The area is $|\mathbf{a} \times \mathbf{b}| = |ab \sin \phi| = 2 \cdot 5 \sin \pi/4 = 5\sqrt{2}$.

3. Given the vectors

$$\mathbf{A} = \mathbf{i}_1 + 2\mathbf{i}_2 + 3\mathbf{i}_3, \qquad \mathbf{B} = 4\mathbf{i}_1 - 5\mathbf{i}_2 - 2\mathbf{i}_3,$$
$$\mathbf{C} = 3\mathbf{i}_1 + 2\mathbf{i}_2 + \mathbf{i}_3, \qquad \mathbf{D} = \mathbf{i}_1 + 3\mathbf{i}_2 + 4\mathbf{i}_3;$$

where $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are an orthonormal basis. Find

(a) $\mathbf{A} - \mathbf{B} + 2\mathbf{C}$;   (Solution (10 points): Answer is $3\mathbf{i}_1 + 11\mathbf{i}_2 + 7\mathbf{i}_3$)

(b) $\mathbf{A} \cdot \mathbf{B}$;   (Solution (10 points): Answer is -12)

(c) The angle made by $\mathbf{C}$ and $\mathbf{D}$;

(Solution (10 points): Answer is

$\cos\phi = \mathbf{C} \cdot \mathbf{D}/(|\mathbf{C}|\ |\mathbf{D}|) = \frac{\sqrt{13}}{2\sqrt{7}}$.

Then find $\phi = 47.04802188$ degrees $= 0.26137790\pi = 0.82114287$.)

(d) The projection of $\mathbf{A}$ onto the direction of $\mathbf{B}$;

(Solution (10 points): $\mathrm{Proj}_{\mathbf{B}}\mathbf{A} = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|^2}\mathbf{B} = -\frac{4}{15}\mathbf{B}$.

(e) The vector product $\mathbf{C} \times \mathbf{D}$.

(Solution (10 points): Use the determinant formula. $\mathbf{C} \times \mathbf{D} = 5\mathbf{i}_1 - 11\mathbf{i}_2 + 7\mathbf{i}_3$.

4. Show that the four vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ given in the previous problem are linearly dependent.

(Proof (10 points): Find nontrivial numbers c_1, c_2, c_3, c_4 such that

$$c_1\mathbf{A} + c_2\mathbf{B} + c_3\mathbf{C} + c_4\mathbf{D} = 0.$$

One can set $c_4 = 1$ and find the other three. Alternatively, one can use theorems from Linear Algebra that says that any 4 vectors in three-dimensional space is linearly dependent.)

5. Use the fomulas learned in class (lecture notes) to verify the following identities:

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= \mathbf{b}[\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})] - \mathbf{a}[\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})] \\ &= \mathbf{c}[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{d})] - \mathbf{d}[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})].\end{aligned}$$

(Proof (10 points): The first line is a direct application of the vector triple product of the form $(\mathbf{a} \times \mathbf{b}) \times \mathbf{E}$ for $\mathbf{E} = \mathbf{c} \times \mathbf{d}$.

For the second line, one uses the vector triple product of the form $\mathbf{A} \times (\mathbf{c} \times \mathbf{d})$ for $\mathbf{A} = \mathbf{a} \times \mathbf{b}$. Then use the scalar triple product formula to find that actually $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{d})$, etc.)