M597K: Solution to Homework Assignment 14 (Last One)

Date: Dec 9, Monday; Due by Monday Dec. 16

1. Use Rayleigh quotient (Section 6.12, Conclusion No. 6) to show that any eigenvalue must be positive ($\lambda > 0$) for Bessel's equation

$$\begin{cases} (ru')' - \frac{m^2}{r}u + \lambda r u = 0, \quad 0 < r < a, \\ u(a) = 0, \\ |u(0)| < \infty, \ |u'(0)| < \infty. \end{cases} \tag{1}$$

(Hint: Once you use the Rayleigh quotient and the boundary conditions, the positivity of λ_n is obvious.)

Solution.

By Rayleigh quotient,

$$\lambda_n = \frac{-p\phi_n\phi_n'|_0^a + \int_0^a (p\phi_n'^2 - q\phi_n^2)dr}{\int_a^b \phi_n^2 \sigma dr}$$

where $p(r) = r$, $q(r) = -\frac{m^2}{r}$, $\sigma(r) = r$. Then

$$\int \phi_n^2 \sigma dr > 0$$

$$\int_0^a (p\phi_n'^2 - q\phi_n^2)dr > 0$$

As $u(a) = 0$, $|u(0)| < \infty$, $|u'(0)| < \infty$,

$$-p\phi_n\phi_n'|_0^a = p(0)\phi_n(0)\phi_n'(0) = 0$$

Then we have $\lambda_n > 0$

2. Use separation of variables to solve the two-dimensional eigenvalue problem for the Laplacian in a rectangle

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0, \quad \text{in } 0 < x < L, \ 0 < y < H, \\ u = 0 \text{ on the boundary.} \end{cases} \tag{2}$$

(The eigenfunctions should be the eigenfunctions of Section 6.9.)

Solution. We use the method of separation of variables.

Let $u = X(x)Y(y)$, we have

$$X''(x)Y(y) + X(x)Y''(y) + \lambda X(x)Y(y) = 0.$$

Then

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \lambda = 0.$$

It must be the case that

$$\frac{X''(x)}{X(x)} = a; \qquad \frac{Y''(y)}{Y(y)} = b,$$

where a and b are constants and $\lambda = -(a+b)$.

By the zero boundary conditions,

$$X(x) = c\sin(\frac{n\pi}{L}x), \qquad a = -(\frac{n\pi}{L})^2,$$

$$Y(y) = c\sin(\frac{m\pi}{H}x), \qquad b = -(\frac{m\pi}{H})^2.$$

Thus we have

$$\lambda_{m,n} = -(a+b) = (\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2,$$

where m and n are integers.

END