M597K: Solutions to Homework Assignment 13

Date: Dec. 16, Monday

1. Use change of variables to reduce

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(t,x), \qquad 0 < x < L, \\ u(0,x) &= g(x), \\ \frac{\partial u}{\partial x}(t,0) &= A(t), \\ \frac{\partial u}{\partial x}(t,L) &= B(t) \end{aligned} \tag{1}$$

to a problem with homogeneous boundary condition.

Solution. Homogeneous boundary condition means that $A(t) = 0$ and $B(t) = 0$. To achieve that, revisit Section 6.10.3. We try

$$V = u - \int_0^x f(t,y)\,dy$$

where $f(t,x) = A(t) + \frac{x}{L}(B(t) - A(t))$ is what was used in that section. Then we see that

$$\frac{\partial V}{\partial x}(t,0) = \frac{\partial V}{\partial x}(t,L) = 0.$$

And

$$V(0,x) = g(x) - xA(0) - \frac{x^2}{2L}(B(0) - A(0)).$$

And using $u(t,x) = V(t,x) + \int_0^x f(t,y)\,dy$ we find an equation for V

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2} + Q(t,x) + \frac{k}{L}[B(t) - A(t)] - [xA'(t) + \frac{x^2}{2L}(B'(t) - A'(t))].$$

Thus the V satisfies the zero boundary condition with a new initial value and a new source term.

2. Find a solution to

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad 0 < x < L,\ 0 < y < H \\ u &= 0 \quad \text{on the boundary of the rectangle,} \\ u(0,x,y) &= 5\sin\left(\frac{7\pi x}{L}\right)\sin\left(\frac{11\pi y}{H}\right) + 3\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{H}\right), \\ \frac{\partial u}{\partial t}(0,x,y) &= 0. \end{aligned}$$

Solution. We use the formula in Section 6.11.2. Since the data $\beta = 0$, we have all $B_{nm} = 0$. Since $\alpha = 5\phi_{7\ 11} + 3\phi_{1\ 1}$, we are able to read off the coefficients in the expansion

$$\alpha(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm}\phi_{nm},$$

by comparison. That is:

$$A_{7\ 11} = 5; \quad A_{1\ 1} = 3.$$

And all others are zero. So the solution is

$$u(t, x, y) = 5\cos(ct\sqrt{(7\pi/L)^2 + (11\pi/H)^2})\phi_{7\ 11} + 3\cos(ct\sqrt{(\pi/L)^2 + (\pi/H)^2})\phi_{1\ 1}.$$

3. Use eigenfunction expansion to derive a solution to

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + Q(t, x, y), \qquad 0 < x < L,\ 0 < y < H \\ u &= 0 \quad \text{on the boundary of the rectangle,} \\ u(0, x, y) &= 0, \\ \frac{\partial u}{\partial t}(0, x, y) &= 0. \end{aligned}$$

Solution. Use eigenfunction expansion

$$u(t, x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm}(t)\phi_{nm}(x, y)$$

where $\phi_{nm}(x, y)$ are the eigenfunctions for the Laplacian on the rectangle with zero boundary condition, see Sect. 6.11.2. Expand the source term in the same way

$$Q(t, x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_{nm}(t)\phi_{nm}(x, y).$$

Then derive an equation for $c_{nm}(t)$.

All the equations for $c_{nm}(t)$ are like this

$$u''(t) + a^2 u(t) = q(t); \quad u(0) = 0, \quad u'(0) = 0. \tag{2}$$

We do the following. First the solutions to

$$u''(t) + a^2 u(t) = 0$$

are

$$u(t) = Ae^{iat}$$

where A is a complex constant. Now let A depend on t and hope that

$$u(t) = A(t)e^{iat} \tag{3}$$

can solve (2). (This is called the method of variation of constants.) So plugging the ansatz (3) back into (2) we find

$$u'' + a^2 u = (A'' + 2aiA')e^{iat} = q(t).$$

So we have the equation for $A(t)$:

$$A''(t) + 2aiA'(t) = e^{-iat}q(t). \tag{4}$$

We can easily see that the initial condition of (2) implies that

$$A(0) = 0, \qquad A'(0) = 0.$$

Now let $A'(t) = B(t)$. From (4) we find that

$$B'(t) + 2aiB(t) = e^{-iat}q(t), \qquad B(0) = 0. \tag{5}$$

We can find B since it is covered in our ODE chapter. Using the integration factor method, we find

$$(B(t)e^{2ait})' = e^{iat}q(t).$$

Thus we find that

$$A'(t) = B(t) = e^{-2ait}\int_0^t e^{ias}q(s)\,ds.$$

Or

$$A(t) = \int_0^t B(\tau)\,d\tau = \int_0^t \int_0^\tau e^{-2ai\tau}e^{ias}q(s)\,ds\,d\tau.$$

We change the order of integration (something we did in the lectures) to find

$$\begin{aligned} A(t) &= \textstyle\int_0^t (\int_s^t e^{-2ai\tau}\,d\tau)e^{ias}q(s)\,ds \\ &= \textstyle\int_0^t \frac{1}{2ai}(e^{-2ais} - e^{-2ait})e^{ias}q(s)\,ds. \end{aligned}$$

Thus we find that

$$\begin{aligned} u(t) = A(t)e^{ait} &= \int_0^t \tfrac{1}{2ai}(e^{ai(t-s)} - e^{-ai(t-s)})q(s)\,ds \\ &= \int_0^t \tfrac{1}{2ai}\{2i\sin[a(t-s)]\}q(s)\,ds \\ &= \int_0^t \tfrac{\sin[a(t-s)]}{a}q(s)\,ds. \end{aligned}$$

Thus we have

$$c_{nm}(t) = \int_0^t \frac{\sin[c(t-s)]}{c} q_{nm}(s)\,ds,$$

where

$$q_{nm}(t) = \frac{4}{LH}\int_0^L \int_0^H Q(t,x,y)\phi_{nm}(x,y)\,dy\,dx,$$

where

$$\phi_{nm}(x,y) = \sin(n\pi x/L)\sin(m\pi y/H).$$

The final solution is

$$u(t,x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}(t)\phi_{nm}(x,y).$$