M597K: Homework Assignment 13 Hint

1. Homogeneous boundary condition means that $A(t) = 0$ and $B(t) = 0$. To achieve that, revisit Section 6.10.3. Try use

$$V = u - \int_0^x f(t, y)\, dy$$

where $f(t, x) = A(t) + \frac{x}{L}(B(t) - A(t))$ is what used in that section.

2. Use the formula in Section 6.11.2. Since your $\beta = 0$, you have all $B_{nm} = 0$. Since $\alpha = 5\phi_{7\ 11} + 3\phi_{1\ 1}$, you should be able to read off the coefficients in the expansion

$$\alpha(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm}\phi_{nm},$$

by comparison. The comparison should be like this: You are given a vector $V = (1, 1, 0)$, and you are told to write it as a linear combination of the three linearly independent vectors $(1, 0, 0), (1, 1, 0), (1, 1, 1)$ as

$$V = \alpha_1(1, 0, 0) + \alpha_2(1, 1, 0) + \alpha_3(1, 1, 1).$$

By comparison we see that $\alpha_1 = 0, \alpha_2 = 1, \alpha_3 = 0$. There is no need to solve a system of equations to find these α's.

3. Use eigenfunction expansion

$$u(t, x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm}(t)\phi_{nm}(x, y)$$

where $\phi_{nm}(x, y)$ are the eigenfunctions for the Laplacian on the rectangle with zero boundary condition, see Sect. 6.11.2. Expand the source term in the same way

$$Q(t, x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_{nm}(t)\phi_{nm}(x, y).$$

Then derive an equation for $c_{nm}(t)$.

When you meet an equation like

$$u''(t) + a^2 u(t) = q(t); \quad u(0) = 0, \quad u'(0) = 0, \tag{1}$$

you can do the following. First the solutions to

$$u''(t) + a^2 u(t) = 0$$

are

$$u(t) = Ae^{iat}$$

where A is a complex constant. Now let A depend on t and hope that

$$u(t) = A(t)e^{iat} \tag{2}$$

can solve (1). (This is called the method of variation of constants.) So plugging the ansatz (2) back into (1) we find

$$u'' + a^2 u = (A'' + 2aiA')e^{iat} = q(t).$$

So we have the equation for $A(t)$:

$$A''(t) + 2aiA'(t) = e^{-iat}q(t). \tag{3}$$

We can easily see that the initial condition of (1) implies that

$$A(0) = 0, \qquad A'(0) = 0.$$

Now let $A'(t) = B(t)$. From (3) we find that

$$B'(t) + 2aiB(t) = e^{-iat}q(t), \qquad B(0) = 0. \tag{4}$$

You can find B since it is covered in our ODE chapter.