# M597K: Solution to Homework Assignment 12

Date: Nov. 25, Monday; Due Wed. Dec. 4.

**1.** Use separation of variables to derive a solution formula for

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \qquad 0 < x < L,\ 0 < y < H \\ u(0, y) &= 0, \\ u(L, y) &= 0, \\ u(x, 0) &= 0, \\ u(x, H) &= g(x). \end{aligned} \tag{1}$$

*Solution.*

Let $u(x, y) = X(x)Y(y)$, the equations become:

$$\frac{Y''(y)}{Y(y)} = -\frac{X''(x)}{X(x)}$$

The left-hand side is a function of $y$, while the right-hand side is a function of $x$. So it must be a constant. Suppose the constant is $\lambda^2$(We can prove that the constant must be positive), we get

$$Y''(y) = \lambda^2 Y(y), \quad X''(x) = -\lambda^2 X(x)$$

The general form of solutions are $X = a_1 e^{i\lambda x} + a_2 e^{-i\lambda x}, \quad Y = b_1 e^{\lambda y} + b_2 e^{-\lambda y}$, where $a_1, a_2, b_1, b_2$ are constants to be determined by boundary conditions.
For this problem $X = a \sin(\frac{n\pi}{L}x)$ which satisfies $X(0) = X(L) = 0$. Similarly, $Y = b \sinh(\frac{n\pi}{L}y)$.
By superposition,

$$u(x, y) = \sum a_n \sin(\frac{n\pi}{L}x) \sinh(\frac{n\pi}{L}y)$$

$$u(x, H) = \sum a_n \sin(\frac{n\pi}{L}x) \sinh(\frac{n\pi}{L}H) = g(x)$$

$$\rightarrow a_n = \frac{2}{L \sinh(\frac{n\pi}{L}H)} \int_0^L g(x) \sin(\frac{n\pi}{L}x) dx$$

so the solution is:

$$u(x, y) = \sum \frac{2}{L \sinh(\frac{n\pi}{L}H)} \sin(\frac{n\pi}{L}x) \sinh(\frac{n\pi}{L}y) \int_0^L g(s) \sin(\frac{n\pi}{L}s)\, ds$$

**2.** By guessing or eigenfunction expansion, find a solution to

$$\begin{aligned} \tfrac{\partial^2 u}{\partial x^2} + \tfrac{\partial^2 u}{\partial y^2} &= 5 \sin \tfrac{3\pi x}{L} \sin \tfrac{2\pi y}{H}, \qquad 0 < x < L,\ 0 < y < H \\ u(0, y) &= 0, \\ u(L, y) &= 0, \\ u(x, 0) &= 0, \\ u(x, H) &= 0. \end{aligned} \tag{2}$$

*Solution.*

First we know the eigenfunctions of $\Delta u = -\lambda u$ are

$$u_{n,m} = \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}, \quad \lambda_{n,m} = (\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2$$

By observation, we see

$$u = -\frac{1}{\lambda_{3,2}} u_{3,2}(x, y) = -\frac{5}{(\frac{3\pi}{L})^2 + (\frac{2\pi}{H})^2} \sin \frac{3\pi x}{L} \sin \frac{2\pi y}{H}$$

End.