M597K: Solution to Homework Assignment 11

Date: Nov. 18, Monday; Due Wed. Nov. 27.

1. Find a solution to

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - au = 0,$$

where a is a constant, with initial condition

$$u(0, x, y) = g(x, y).$$

Hint: Show that $v = e^{-at}u$ satisfies the standard heat equation.

Solution.

$$\begin{cases} \frac{\partial u}{\partial t} - (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) - au = 0 \\ \\ u(0, x, y) = g(x, y) \end{cases}$$

Let

$$v(t, x, y) = e^{-at}u(t, x, y)$$

then we have

$$\frac{\partial v}{\partial t} = e^{-at}\frac{\partial u}{\partial t} - ae^{-at}u$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = e^{-at}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

With the boundary condition: $v(0, x, y) = e^0 u(0, x, y) = g(x, y)$ the problem reduced to

$$\begin{cases} \frac{\partial v}{\partial t} - (\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) = 0 \\ \\ v(0, x, y) = g(x, y) \end{cases}$$

which is a standard heat equation.

The solution is :

$$v(t, x, y) = \frac{1}{4\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\lambda, \mu)e^{-\frac{(x-\lambda)^2+(y-\mu)^2}{4t}}d\lambda d\mu$$

so

$$u(t, x, y) = \frac{e^{at}}{4\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\lambda, \mu)e^{-\frac{(x-\lambda)^2+(y-\mu)^2}{4t}}d\lambda d\mu$$

2. Find a solution $u(t,x)$ to

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0, \quad t>0,\ x>0, \\ u(0,x) &= g(x), \quad x>0, (g(0)=0) \\ u(t,0) &= 0, \quad t>0. \end{cases} \tag{1}$$

Hint: Consider the odd extension of the initial data $g(x)$:

$$G(x) = \begin{cases} g(x), & x>0, \\ -g(-x), & x<0, \end{cases} \tag{2}$$

and solve the heat equation in the whole line with initial data $G(x)$.

Solution.

Let

$$G(x) = \begin{cases} g(x) & x>0 \\ 0 & x=0 \\ -g(-x) & x<0 \end{cases}$$

Solve the equations:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 & t>0, x\in \mathbf{R} \\ u(0,x) = G(x) & x \in \mathbf{R} \\ u(t,0) = 0 & t>0 \end{cases}$$

to get the solution

$$u(t,x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} G(y) e^{-\frac{(x-y)^2}{4t}} dy$$

Restricting $x>0$ we get the solution of original problem.

Note:why do we choose odd extension not even extension?

To ensure in the solution $u(t,0)=0$. Please check it.

3. Find the fundamental solution ϕ for a dipole source:

$$\Delta\phi(\mathbf{x}) = \nabla\delta(\mathbf{x})\cdot\mathbf{y} \equiv \lim_{h\to 0} \frac{\delta(\mathbf{x}+h\mathbf{y}) - \delta(\mathbf{x})}{h},$$

where $\mathbf{y}$ is a fixed unit vector. (In electrostatics, a dipole source is like a battery with its two terminals extremely close together in physical space.)

Hint: See Keener (text book), p. 342.

Solution.

Recall:

$$\triangle(-\frac{1}{4\pi|\mathbf{x}|}) = \delta(\mathbf{x}), \quad \triangle(-\frac{1}{4\pi|\mathbf{x}-\mathbf{x_0}|}) = \delta(\mathbf{x}-\mathbf{x_0})$$

so the solution of the problem is

$$\Phi(\mathbf{x}) = \lim_{h\to 0}\frac{1}{4\pi h}(\frac{1}{|\mathbf{x}|} - \frac{1}{|\mathbf{x}+h\mathbf{y}|}) = \frac{-1}{4\pi}\nabla(\frac{1}{|\mathbf{x}|})\cdot\mathbf{y} = \frac{\mathbf{x}\cdot\mathbf{y}}{4\pi|\mathbf{x}|^3}$$

Note:

$$\nabla\frac{1}{|\mathbf{r}|} = \frac{-\mathbf{r}}{|\mathbf{r}|^3}$$

Please check it.

Optional problem. Apply Duhamel's principle to derive a solution to

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) = f(t, x_1, x_2, x_3),$$

with initial condition

$$u(0, x_1, x_2, x_3) = 0.$$

The idea is similar to that of the wave equation.

Solution.

$$u(t,\mathbf{x}) = \int_0^t \frac{1}{[4\pi(t-\tau)]^{\frac{3}{2}}}(\int_{\mathbf{R}^3} f(\tau,\mathbf{y})e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4(t-\tau)}}\,d\mathbf{y})\,d\tau$$

End.