1. Find a solution to
\[ \frac{\partial u}{\partial t} - \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - au = 0, \]
where \( a \) is a constant, with initial condition
\[ u(0, x, y) = g(x, y). \]

*Hint:* Show that \( v = e^{-at}u \) satisfies the standard heat equation.

2. Find a solution \( u(t, x) \) to
\[
\begin{cases}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, & t > 0, \quad x > 0, \\
u(0, x) = g(x), & x > 0, \quad (g(0) = 0) \\
u(t, 0) = 0, & t > 0.
\end{cases}
\]

*Hint:* Consider the odd extension of the initial data \( g(x) \):
\[
G(x) = \begin{cases} 
  g(x), & x > 0, \\
  -g(-x), & x < 0, 
\end{cases}
\]
and solve the heat equation in the whole line with initial data \( G(x) \).

3. Find the fundamental solution \( \phi \) for a dipole source:
\[ \Delta \phi(x) = \nabla \delta(x) \cdot \mathbf{y} \equiv \lim_{h \to 0} \frac{\delta(x + h\mathbf{y}) - \delta(x)}{h}, \]
where \( \mathbf{y} \) is a fixed unit vector. (In electrostatics, a dipole source is like a battery with its two terminals extremely close together in physical space.)


**Optional problem.** Apply Duhamel’s principle to derive a solution to
\[ \frac{\partial u}{\partial t} \left( \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \right) = f(t, x_1, x_2, x_3), \]
with initial condition
\[ u(0, x_1, x_2, x_3) = 0. \]
The idea is similar to that of the wave equation.

End.