# M597K: Solution to Homework Assignment 10

Date: Nov. 11, Monday; Due Wed. Nov. 20.

**1.** Use the method of characteristics to derive a solution formula to

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + cu = 0, \qquad u(0,x) = g(x),$$

where $a$ and $c$ are constants, $t > 0$, and $x \in I\!R^1$. (Hint: Derive an ordinary differential equation for $u$ along a characteristic curve $x = x_0 + at$.)

*Solution.* The characteristic line is given by the equation

$$x'(t) = a$$

The solutions are $x(t) = at + x_0$

where $x_0$ is a constant depending on $x$

Along the characteristic line the equation is

$$\frac{du}{dt} = -cu$$

$$u = we^{-ct}$$

where $w$ is a constant

Determine the constants by the initial value

$$u(0, x(0)) = w = g(x(0)) = g(x_0)$$

Replace $x_0 = x - ct$ we get solution

$$u(t,x) = g(x - at)e^{-ct}.$$

**2.** Use the appropriate formula to find the solution $u(t,x)$ to

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} & = 0, \\ u(0,x) & = e^{-x^2}, \\ \frac{\partial u}{\partial t}(0,x) & = 0. \end{cases}$$

*Solution.*

According to D'Alenbert formula:

$c = 2,\ g(x) = e^{-x^2},\ h(x) = 0,\ f(t,x) = 0$

Therefore the solution is:

$$u(t,x) = \frac{1}{2}[e^{-(x+2t)^2} + e^{-(x-2t)^2}] + \frac{1}{4}\int_{x-2t}^{x+2t} h(y)dy + \frac{1}{4}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)dyds$$

$$u(t,x) = \frac{1}{2}[e^{-(x+2t)^2} + e^{-(x-2t)^2}].$$

**3.** Find the value at $x = 0, t = 2$ of the solution of the initial value problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= x, \\ u(0,x) &= 0, \\ \frac{\partial u}{\partial t}(0,x) &= 0. \end{cases}$$

*Solution.*

According to D'Alenbert formula:

$c = 1,\ g(x) = 0,\ h(x) = 0,\ f(t,x) = x$

Therefore the solution is:

$$u(t,x) = \frac{1}{2}[g(x+ct) + g(x-ct)] + \frac{1}{2}\int_{x-2t}^{x+2t} h(y)dy + \frac{1}{2}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)dyds$$

$$u(t,x) = \frac{1}{2}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)dyds$$

$$u(t,x) = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} ydyds$$

$$u(t,x) = \frac{1}{2}\int_0^t \frac{1}{2}y^2|_{x-(t-s)}^{x+(t-s)}ds$$

$$u(t,x) = \frac{1}{2}\int_0^t \frac{1}{2}[(x+(t-s))^2 - (x+(t-s))^2]ds$$

$$u(t,x) = \int_0^t x(t-s)ds = xt^2 - \frac{1}{2}xt^2 = \frac{1}{2}xt^2$$

If $x = 0,\ t = 2,\ u(2,0) = 0$.

**4.** In the three dimensional homogeneous wave equation ($f = 0$) with speed $c = 1$, let $u(0, x_1, x_2, x_3) = 0$, $\frac{\partial u}{\partial t}(0, x_1, x_2, x_3) = h(x_1, x_2, x_3)$. Suppose that $h(x_1, x_2, x_3) = 0$ for $x_1^2 + x_2^2 + x_3^2 \geq 1$. Show or explain that $u(t, 0, 0, 0) = 0$ for all $t > 1$.

*Solution.*

Using formula

$$u(t, x_1, x_2, x_3) = \frac{t}{4\pi(ct)^2} \int\int_{|\mathbf{y}-\mathbf{x}|=ct} h(\mathbf{y}) dS_y$$

For this problem

$$u(t, 0, 0, 0) = \frac{t}{4\pi t^2} \int\int_{|\mathbf{y}|=t} h(\mathbf{y}) dS_y$$

for $t > 1, \quad h(\mathbf{y}) = 0$, for all $|\mathbf{y}| = t$ so the integration is 0.

**5.** In the two dimensional homogeneous wave equation ($f = 0$) with speed $c = 1$, let $u(0, x_1, x_2) = 0$, $\frac{\partial u}{\partial t}(0, x_1, x_2) = h(x_1, x_2)$. Suppose that $h(x_1, x_2) = 0$ for $x_1^2 + x_2^2 \geq 1$ and $h(x_1, x_2) > 0$ for $x_1^2 + x_2^2 < 1$. Show or explain that $u(t, 0, 0) > 0$ for all $t > 0$.

*Solution.*

Using formula

$$u(t, x_1, x_2) = \frac{1}{2\pi} \int\int_{r<t} \frac{h(y_1, y_2)}{\sqrt{t^2 - r^2}} dy_1 dy_2$$

For this problem

$$u(t, 0, 0) = \frac{1}{2\pi} \int\int_{r<t} \frac{h(y_1, y_2)}{\sqrt{t^2 - r^2}} dy_1 dy_2$$

since for $t > 1, \quad h(\mathbf{y}) = 0$, for all $|\mathbf{y}| = t$
the integral equals to

$$\frac{1}{2\pi} \int\int_{r< \min\ (t,1)} \frac{h(y_1, y_2)}{\sqrt{t^2 - r^2}} dy_1 dy_2$$

since $h(\mathbf{y}) > 0$, for $|\mathbf{y}| < 1$ the integral is positive.

**Optional 6.** Use the method of characteristics to solve the problem

$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad t > 0,\ x \in I\!\!R, \\ u(0, x) = e^x. \end{cases}$$

*Solution.*

Characteristic line $x'(t) = u$ , along the characteristic line the equation is $u'(t) = 0$. Thus $u(t) = e^{x_0}$.

$x'(t) = u = e^{x_0}$, $x(t) = e^{x_0}t + C$.

As $x(0) = x_0$, we get $C = x_0$, $x = e^{x_0}t + x_0$.

As $u = e^{x_0}$, we get $x = ut + \ln u$

$u(x,t)$ is the solution of $ut + \ln u - x = 0$, as there is no other way to express this, we can only write it in this way.

====End====