

## 6.10. Heat equation in an interval.

### 6.10.1. Initial Value Problem

Consider

$$\begin{cases} \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, & 0 < x < L, \\ u(t, 0) = u(t, L) = 0, \\ u(0, x) = g(x). \end{cases} \quad (1)$$

We try separation of variables:

$$u(t, x) = \phi(x)G(t). \quad (2)$$

Then

$$\frac{G'(t)}{kG(t)} = \frac{\phi''(x)}{\phi(x)}.$$

Thus we set them to be a common constant  $-\lambda$ :

$$G'(t) = -\lambda k G(t), \quad (3)$$

$$\phi''(x) = -\lambda \phi(x). \quad (4)$$

For  $\phi$  satisfying

$$\phi(0) = \phi(L) = 0, \quad (5)$$

we find the solutions to the **eigenvalue problem**(4)-(5):

$$\phi(x) = c \sin\left(\frac{n\pi x}{L}\right), \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \dots. \quad (6)$$

So we have solutions

$$u(t, x) = \sum_{n=1}^{\infty} C_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right). \quad (7)$$

We need (7) to satisfy the initial condition

$$u(0, x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = g(x).$$

By Fourier sine series, we only need to take

$$C_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

So a complete solution to (1) is

$$u(t, x) = \sum_{n=1}^{\infty} \left( \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx \right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right).$$

### 6.10.2. Inhomogeneous Problem

Consider

$$\begin{cases} \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q(t, x), & 0 < x < L, \\ u(t, 0) = u(t, L) = 0, \\ u(0, x) = 0. \end{cases} \quad (8)$$

Let us use the eigenfunctions and propose a solution in the form

$$u(t, x) = \sum_{n=1}^{\infty} C_n(t) \sin\left(\frac{n\pi x}{L}\right), \quad C_n(0) = 0.$$

We expand  $Q(t, x)$  as

$$Q(t, x) = \sum_{n=1}^{\infty} q_n(t) \sin\left(\frac{n\pi x}{L}\right).$$

We note that  $Q(t, x)$  may not have zero value at the boundaries, but the expansion is still valid in the  $L^2(0, L)$  sense. Then (8) can be projected onto the component  $\sin\left(\frac{n\pi x}{L}\right)$ :

$$C_n'(t) = -k\left(\frac{n\pi}{L}\right)^2 C_n + q_n(t).$$

We find

$$C_n(t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \int_0^t e^{k\left(\frac{n\pi}{L}\right)^2 s} q_n(s) ds.$$

Thus a solution to (8) is

$$u(t, x) = \sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \left( \int_0^t e^{k\left(\frac{n\pi}{L}\right)^2 s} q_n(s) ds \right) \sin\left(\frac{n\pi x}{L}\right),$$

where

$$q_n(t) = \frac{2}{L} \int_0^L Q(t, x) \sin\left(\frac{n\pi x}{L}\right) dx. \quad (n = 1, 2, \dots)$$

### 6.10.3. Boundary Value Problem

Consider

$$\begin{cases} \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, & 0 < x < L, \\ u(0, x) = 0, \\ u(t, 0) = a(t), \\ u(t, L) = b(t). \end{cases} \quad (9)$$

We use the variable

$$V = u - [a(t) + \frac{x}{L}(b(t) - a(t))].$$

Then

$$\begin{cases} \frac{\partial V}{\partial t} = k \frac{\partial^2 V}{\partial x^2} - [a'(t) + \frac{x}{L}(b'(t) - a'(t))], & 0 < x < L, \\ V(0, x) = -[a(0) + \frac{x}{L}(b(0) - a(0))], \\ V(t, 0) = 0, \\ V(t, L) = 0. \end{cases}$$

This can be solved by the previous two steps.

We shall solve the heat equation in a two-dimensional rectangular domain off the class, see the lecture notes.

#### 6.10.4. Two-dimensional rectangular domains.

Consider

$$\begin{cases} \frac{\partial u}{\partial t} = k \Delta u, & 0 < x < L, \quad 0 < y < H, \\ u(0, x, y) = g(x, y), \\ u(t, x, y) = 0 & \text{on the boundary of the rectangle.} \end{cases} \quad (10)$$

We use eigenfunction expansion

$$u(t, x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{mn}(t) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right).$$

Then the initial condition

$$g(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{mn}(0) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)$$

implies that

$$C_{mn}(0) = \frac{4}{LH} \int_0^L \int_0^H g(x, y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) dx dy \equiv g_{mn}.$$

The equation implies

$$C'_{mn}(t) = -k\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]C_{mn}.$$

Thus

$$C_{mn}(t) = g_{mn} e^{-k\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]t}.$$

Hence a solution is

$$u(t, x, y) = \sum_{n,m=1}^{\infty} g_{mn} e^{-k\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]t} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right).$$