6.9. Poisson equation in a rectangle.

We consider

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \quad 0 < x < L, \quad 0 < y < H, \tag{1}$$

$$u(0,y) = u(L,y) = 0, \quad 0 < y < H, \tag{2}$$

$$u(x,0) = u(x,H) = 0, \quad 0 < x < L. \tag{3}$$

We propose the eigenvalue problem

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= -\lambda u, \quad 0 < x < L, \quad 0 < y < H, \\ u(0,y) &= u(L,y) = 0, \quad 0 < y < H, \\ u(x,0) &= u(x,H) = 0, \quad 0 < x < L. \end{cases} \tag{4}$$

We use separation of variables to find

$$u(x,y) = \sin(\frac{n\pi x}{L})\sin(\frac{m\pi y}{H}), \quad \lambda = (\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2, \tag{5}$$

where $n = 1, 2, \cdots, m = 1, 2, \cdots$, are solutions to (4). Using Fourier sine series Theorem in the $x-$variable first and then in the y-variable, we can expand $f(x,y)$ into

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm} \sin(\frac{n\pi x}{L})\sin(\frac{m\pi y}{H}), \tag{6}$$

where

$$B_{nm} = \frac{4}{LH}\int_0^L \int_0^H f(x,y)\sin(\frac{n\pi x}{L})\sin(\frac{m\pi y}{H})\, dydx. \tag{7}$$

Thus a solution to (1)-(3) is

$$u(x,y) = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{B_{nm}}{(\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2} \sin(\frac{n\pi x}{L})\sin(\frac{m\pi y}{H}). \tag{8}$$

where B_{nm} are given in (7). We can verify that (8) is indeed a solution:

$$\begin{aligned} \triangle u &= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{B_{nm}}{(\frac{n\pi}{L})^2+(\frac{m\pi}{H})^2} \triangle(\sin(\frac{n\pi x}{L})\sin(\frac{m\pi y}{H})) \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm} \sin(\frac{n\pi x}{L})\sin(\frac{m\pi y}{H}) \\ &= f(x,y). \end{aligned}$$

Note: Let Ω be a general domain in $I\!R^n$. Consider

$$\begin{cases} -\triangle u &= \lambda u, \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{9}$$

Then in general there is no explicit formula for u or λ. But a general theorem tells us that there exists $\lambda_1 < \lambda_2 < \cdots$ and corresponding $u_1, u_2 \cdots$ such that (λ_j, u_j) are solutions to the **eigenvalue problem** (9) and any$f(\vec{x})$ can be written as

$$f(\vec{x}) = \sum_{n=1}^{\infty} B_n u_n. \tag{10}$$

Thus a solution to

$$\triangle u = f(x)$$

is

$$u = -\sum_{n=1}^{\infty} \frac{B_n}{\lambda_n} u_n. \tag{11}$$

So the study of the eigenvalue problem is very useful. The expansion (10) is called the **eigen-function expansion**. The equation

$$-\triangle u = \lambda u \tag{12}$$

is called the **Helmholtz equation**.

The eigen functions u_n are orthogonal in the sense

$$\int_\Omega u_n(x) u_m(x)\, dx = 0, \qquad \text{if } n \neq m. \tag{13}$$

This is why we can determine the coefficients B_n quickly:

$$B_n = \frac{\int_\Omega f(x) u_n(x)\, dx}{\int_\Omega (u_n(x))^2\, dx} \tag{14}$$

for all $n = 1, 2, \cdots$.