6.6. (Continued)

We have found a solution to

$$\triangle u = f(x) \quad \text{in} \quad I\!R^3$$

in the form

$$u(x) = -\frac{1}{4\pi}\int\int\int_{I\!R^3} \frac{f(y)}{|x-y|}dy$$

via the Fourier transform, but we were not able to invert the function $\frac{1}{|\omega|^2}$ in class. We started another approach: Finding a radial solution $A(x) = A(|x|)$ to

$$\triangle A = \delta(x).$$

The solution is called the fundamental solution, and it has the form $A(x) = -\frac{1}{4\pi|x|}$ in three-dimensions. We now derive the fundamental solution via a new approach. Recall in $I\!R^n$ (Chapter 1, Section 1.16.2)

$$\triangle A(r) = \frac{\partial^2 A}{\partial r^2} + \frac{n-1}{r}\frac{\partial A}{\partial r}.$$

At $r \neq 0$, we have

$$\frac{\partial^2 A}{\partial r^2} + \frac{n-1}{r}\frac{\partial A}{\partial r} = 0. \tag{1}$$

In $I\!R^3$, we can integrate the above equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}[r^2\frac{\partial A}{\partial r}] = 0.$$

Then

$$\begin{aligned} r^2\tfrac{\partial A}{\partial r} &= C, \\ \tfrac{\partial A}{\partial r} &= \tfrac{C}{r^2}, \\ A &= C_1 - \tfrac{C}{r}. \end{aligned}$$

The constant C_1 will be arbitrary. But the constant C is determined by the strength of the $\delta(x)$:

$$\int_{|x|<1} \triangle A dx = \int \delta(x)\, dx = 1.$$

A calculation can be done to find (skipped in class, but see Keener p.341, or later of this lecture notes):

$$C = \frac{1}{4\pi}.$$

Thus

$$A(r) = -\frac{1}{4\pi r} + C_1.$$

This function

$$A(r) = -\frac{1}{4\pi r}$$

is called the **fundamental solution** to the Laplace equation in $I\!R^3$.

The constant C. We find the constant C in $A(r)$ here. We use

$$f_\varepsilon(r) = \begin{cases} \frac{1}{\frac{4}{3}\pi\varepsilon^3}, & r < \varepsilon, \\ 0, & r > \varepsilon \end{cases}$$

to approximate the $\delta(x)$. Then equation (1) is

$$\frac{1}{r^2}\frac{\partial}{\partial r}[r^2\frac{\partial A}{\partial r}] = f_\varepsilon(r).$$

We integrate and use the condition $r^2\frac{\partial A}{\partial r}|_{r=0} = 0$ to find

$$\begin{aligned} r^2\frac{\partial A}{\partial r} - 0 &= \int_0^r r^2 f_\varepsilon(r)dr \\ &= \begin{cases} \frac{1}{\frac{4}{3}\pi\varepsilon^3}\frac{r^3}{3}, & r < \varepsilon, \\ \frac{1}{4\pi}, & r \geq \varepsilon. \end{cases} \end{aligned}$$

Let $\varepsilon \to 0$, we conclude that

$$\begin{aligned} r^2\frac{\partial A}{\partial r} &= \frac{1}{4\pi}, \quad r \geq 0, \\ \frac{\partial A}{\partial r} &= \frac{1}{4\pi r^2}, \\ A &= -\frac{1}{4\pi r} + C_1. \end{aligned}$$

We find

$$A(x) = -\frac{1}{4\pi|x|} + C_1.$$

We call

$$A(x) = -\frac{1}{4\pi|x|}$$

the fundamental solution to the Laplacian in $I\!R^3$.

In $I\!R^2$: We follow the same idea in $I\!R^3$. We consider a radial solution $A(|x|)$ to

$$\triangle A = \delta(x) \quad \text{in} \quad I\!R^2. \tag{2}$$

We have known that

$$\triangle A = \frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r}$$

in $I\!R^2$. Thus

$$\triangle A = \frac{1}{r}\frac{\partial}{\partial r}[r\frac{\partial A}{\partial r}].$$

At $r \neq 0$, equation (2) is

$$\frac{1}{r}\frac{\partial}{\partial r}[r\frac{\partial A}{\partial r}] = 0.$$

So

$$r\frac{\partial A}{\partial r} = C.$$

Or

$$A = C \ln r + C_1.$$

We need to find the constant C; the other constant C_1 is not important and we set it to zero. To do so, we approximate $\delta(x)$ in $I\!R^3$ by

$$F_\varepsilon(x) = \begin{cases} \frac{1}{\pi\varepsilon^2}, & |x| < \varepsilon, \\ 0, & |x| \geq \varepsilon \end{cases}$$

and consider radial solution to

$$\triangle A(x) = F_\varepsilon(x).$$

We find

$$\frac{1}{r}\frac{\partial}{\partial r}[r\frac{\partial A}{\partial r}] = F_\varepsilon(r)$$

Or

$$\tfrac{\partial}{\partial r}[r\tfrac{\partial A}{\partial r}] = rF_\varepsilon(r). \tag{3}$$

Since $A(|x|)$ is radial, we expect and assume $\frac{\partial A}{\partial r} = 0$ at $r = 0$. Then we integrate (3) in r over $[0, r]$ to find

$$r\frac{\partial A}{\partial r} = \int_0^r sF_\varepsilon(s)ds = \begin{cases} \frac{1}{\pi\varepsilon^2}(\frac{r^2}{2}), & r < \varepsilon, \\ \frac{1}{2\pi}, & r \geq \varepsilon \end{cases}$$

Now let $\varepsilon \to 0$, we find

$$r\frac{\partial A}{\partial r} = \frac{1}{2\pi}, \quad r \geq 0.$$

Thus

$$A = \frac{1}{2\pi} \ln r + C_1.$$

As usual, we set $C_1 = 0$. By Superposition we find that a solution to

$$\triangle u = f$$

in $I\!R^2$ is

$$u(x) = A(x) * f(x) = \frac{1}{2\pi} \int_{I\!R^2} \ln |x - x_0| f(x_0) dx_0.$$

Application: Biot-Savart law (Keener, p.342)

Consider an incompressible fluid with velocity $\vec{u}$:

$$\text{div } \vec{u} = 0$$

and vorticity

$$\vec{\omega} = \text{curl } \vec{u}$$

in $I\!R^3$. Introduce a vector potential $\vec{A}$ (whose existence follows from div $\vec{u} = 0$) such that

$$\vec{u} = \text{curl } \vec{A}$$

and

$$\text{div } \vec{A} = 0.$$

Then

$$\begin{aligned} \vec{\omega} &= \text{curl } (\text{curl } \vec{A}) \\ &= \nabla(\text{div } \vec{A}) - \text{div } (\nabla \vec{A}) \\ &= -\triangle \vec{A}, \end{aligned}$$

Using the solution formula for Poisson equation, we have

$$\vec{A} = \frac{1}{4\pi} \int \frac{\omega(\vec{y})}{|x - y|} dy.$$

Thus

$$\begin{aligned} u(\vec{x}) &= \text{curl } \vec{A} \\ &= \tfrac{1}{4\pi} \int \text{curl } (\tfrac{\vec{\omega}(y)}{|x-y|}) dy \\ &= \tfrac{1}{4\pi} \int \text{curl }_x (\tfrac{1}{|x-y|}) \times \vec{\omega}(y) dy \\ &= \tfrac{1}{4\pi} \int \tfrac{(\vec{x}-\vec{y}) \times \vec{\omega}(\vec{y})}{|\vec{x}-\vec{y}|^3} d\vec{y}. \end{aligned}$$

This is the **Biot-Savart Law** in $I\!R^3$, which relates vorticity $\vec{\omega}$ to velocity $\vec{u}$.

Biot-Savart law in $I\!R^2$.

Consider an imcompressible fluid in $I\!R^2$ with velocity $\vec{u}$:

$$\text{div } \vec{u} = 0.$$

There exists a **stream function** (a scalar function) ψ such that

$$\vec{u} = (u_1, u_2) = (\partial_{x_2}\psi, -\partial_{x_1}\psi).$$

The vorticity of $\vec{u}$ will be a scalar ω, which is the third component of the physical vorticity $\vec{\omega}$. We have

$$\begin{aligned}\omega &= \text{curl } \vec{u} \\ &= \partial_{x_1}u_2 - \partial_{x_2}u_1 \\ &= -\partial_{x_1}\partial_{x_1}\psi - \partial_{x_2}\partial_{x_2}\psi \\ &= -\triangle\psi.\end{aligned}$$

Thus

$$\psi = -\frac{1}{2\pi}\int_{I\!R^2} \ln|x-y|\omega(y)dy.$$

Hence

$$\begin{aligned}\vec{u} &= -\frac{1}{2\pi}\int_{I\!R^2} \frac{(x_2-y_2, y_1-x_1)}{|\vec{x}-\vec{y}|^2}\omega(\vec{y})d\vec{y} \\ &= -\frac{1}{2\pi}\int_{I\!R^2} \frac{(y_2,-y_1)}{|\vec{y}|^2}\omega(\vec{x}-\vec{y})d\vec{y} \\ &= \frac{1}{2\pi}\int_{I\!R^2} \frac{\vec{y}^{\perp}}{|\vec{y}|^2}\omega(\vec{x}-\vec{y})d\vec{y},\end{aligned}$$

where $\vec{y}^{\perp} = (-y_2, y_1)$. See Figure 6.6.1 for $\vec{y}^{\perp}$, which is a rotation by 90° counter-clockwise of $\vec{y}$.

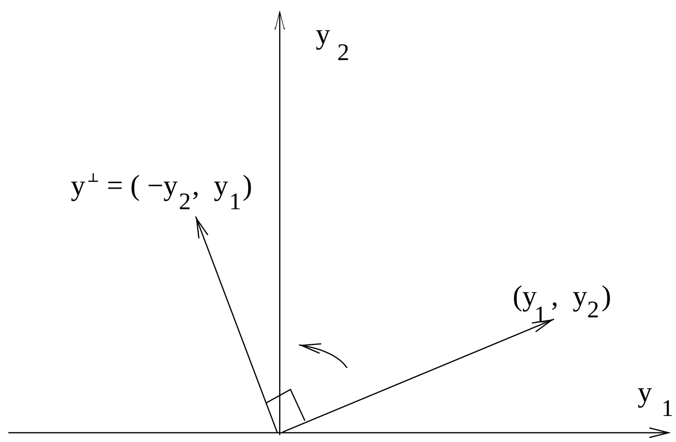

Figure 6.7.1. The relation between the perp of y to y.

6.7 Concept of Fundamental Solutions.

1. The solution to

$$\triangle u = \delta(x) \quad \text{in} \quad I\!R^n$$

is called the fundamental solution to the Laplace equation. This $\delta(x)$ may be regarded as a point charge.

2. The solution to

$$\begin{cases} \frac{\partial}{\partial t} u - \triangle u &= 0, \\ u(0,x) &= \delta(x), \end{cases}$$

which is

$$F(t,x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{x^2}{4t}}$$

is called the fundamental solution to the heat equation. This $\delta(x)$ may be regarded as a point spot of heat.