6.5. Heat equation in $\mathbb{R}^n$ and $\mathbb{R}^1_+$.

Modeling heat conduction. (Keener, p.380.)

We propose to study heat conduction in a material, for example, gas or metal. Let $u(t,\vec{x})$ be temperature. Then the total thermal energy in a region Ω is

$$\int_\Omega \rho\, c\, u(t,\vec{x})\, d\vec{x},$$

where $\rho(t,\vec{x})$ is density of mass, c is heat capacity (energy/unit mass). Let $\vec{q}$ be heat flux: energy per unit area per unit time; let $f(t,\vec{x})$ be heat production: energy per unit volume per unit time. Then "*conservation of energy*" is

$$\frac{d}{dt}\int_\Omega \rho c u(t,\vec{x}) d\vec{x} = \int_\Omega f(t,\vec{x})\, d\vec{x} - \int_{\partial\Omega} \vec{q}\cdot\vec{n}\, dS,$$

where $\vec{n}$ is the unit outward normal to the boundary $\partial\Omega$. Through physical experiments, there holds *Fourier's law of cooling*:

$$\vec{q} = -k\nabla u$$

for many common materials. By Gauss divergence theorem, see Chapter 1.6, we thus have

$$\int_\Omega [\frac{\partial}{\partial t}(\rho c u) - f] d\vec{x} = k\int_{\partial\Omega} \nabla u \cdot \vec{n} dS = k\int_\Omega \text{div } (\nabla u) d\vec{x}.$$

Or

$$\int_\Omega [\frac{\partial}{\partial t}(\rho c u) - k\,\text{div } (\nabla u) - f] d\vec{x} = 0.$$

for all Ω. Thus

$$\frac{\partial}{\partial t}(\rho c u) = k\,\text{div } (\nabla u) - f.$$

Assuming ρ =constant, c =constant. Let $D = \frac{k}{\rho c}$ (the diffusion coefficient). Then

$$\frac{\partial}{\partial t} u = D\triangle u + \frac{f}{\rho c},$$

where

$$\triangle = \text{div } \nabla = \sum_{i=1}^n \partial_i^{\,2}$$

is called the Laplacian. We can use $\tau = Dt$ so that the scaled equation is

$$\frac{\partial u}{\partial t} = \triangle u + \frac{f}{\rho c D}.$$

Solution. Consider

$$\begin{cases} \frac{\partial}{\partial t}u &= \triangle u \\ u(0,\vec{x}) &= g(\vec{x}). \end{cases}$$

Use Fourier transform in $I\!R^n$:

$$\hat{u}(t,\vec{\omega}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{I\!R^n} u(t,\vec{x}) e^{i\omega\cdot\vec{x}} d\vec{x}.$$

Then

$$\begin{cases} \frac{\partial}{\partial t}\hat{u} &= -(\omega_1^2 + \cdots + \omega_n^2)\hat{u}, \\ \hat{u}(0,\vec{\omega}) &= \hat{g}. \end{cases}$$

Thus

$$\hat{u} = \hat{g} e^{-(\omega_1^2+\cdots+\omega_n^2)t} = \hat{g}[\frac{1}{(2t)^{\frac{n}{2}}} e^{-\frac{x_1^2+\cdots+x_n^2}{4t}}]^{\wedge}.$$

By inversion theorem,

$$\begin{aligned} u &= \hat{u}^{\vee} = \tfrac{1}{(2\pi)^{\frac{n}{2}}} g * \tfrac{1}{(2t)^{\frac{n}{2}}} e^{-\frac{x_1^2+\cdots+x_n^2}{4t}} \\ &= \tfrac{1}{(4\pi t)^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(y_1,\cdots,y_n) e^{-\frac{(x_1-y_1)^2+\cdots-(x_n-y_n)^2}{4t}} dy_1\, dy_2 \cdots dy_n. \end{aligned} \tag{1}$$

Theorem 6.5.1. A solution to

$$\partial_t u = k\triangle u, \quad u(0,x) = g(x), \quad x \in I\!R^n$$

is

$$u(t,x) = \frac{1}{(4\pi k t)^{\frac{n}{2}}} \int_{I\!R^n} g(y) e^{-\frac{|x-y|^2}{4kt}} dy.$$

Theorem 6.5.2. A solution to

$$\begin{cases} \frac{\partial}{\partial t}u &= k\triangle u + f(t,x), \\ u(0,\vec{x}) &= 0 \end{cases}$$

is

$$u(t,x) = \int_0^t \frac{1}{(4\pi k(t-\tau))^{\frac{n}{2}}} \int_{I\!R^n} f(\tau,y) e^{-\frac{|x-y|^2}{4k(t-\tau)}} dy\, d\tau.$$

(Duhamel's principle).

The heat equation in $I\!R^1_+$ is a homework problem.

Notes 1. Multi-dimensional Fourier transform is equivalent to one-dimensional Fourier transform applied many times. For example, the two-dimensional Fourier transform is

$$\begin{aligned}
\hat{u}(\omega_1, \omega_2) &= \tfrac{1}{2\pi} \int u(x_1, x_2) e^{ix_1\omega_1 + ix_2\omega_2} dx_1 dx_2 \\
&= \tfrac{1}{(2\pi)^{\frac{1}{2}}} \int [\tfrac{1}{(2\pi)^{\frac{1}{2}}} \int u(x_1, x_2) e^{i\omega_2 x_2} dx_2] e^{i\omega_1 x_1} dx_1 \\
&= \tfrac{1}{(2\pi)^{\frac{1}{2}}} \int \hat{u}(x_1, \omega_2) e^{i\omega_1 x_1} dx_1 = (\hat{u}(x_1, \omega_2))^\wedge(\omega_1, \omega_2).
\end{aligned} \tag{2}$$

2. Transform formula: It follows easily from the one-dimensional one:

$$(e^{-\beta(x_1^2+x_2^2+\cdots+x_n^2)})^\wedge = (e^{-\beta x_1^2})^\wedge (e^{-\beta x_2^2})^\wedge \cdots (e^{-\beta x_n^2})^\wedge$$

$$= \frac{1}{(2\beta)^{\frac{1}{2}}} e^{-\frac{\omega_1^2}{4\beta}} \frac{1}{(2\beta)^{\frac{1}{2}}} e^{-\frac{\omega_2^2}{4\beta}} \cdots \frac{1}{(2\beta)^{\frac{1}{2}}} e^{-\frac{\omega_n^2}{4\beta}} = \frac{1}{(2\beta)^{\frac{n}{2}}} e^{-\frac{1}{4\beta}(\omega_1^2+\omega_2^2+\cdots+\omega_n^2)}.$$

3. Inverse transform: Definition:

$$u^\vee = (2\pi)^{-\frac{n}{2}} \int_{I\!R^n} u(x) e^{-i\omega\cdot x} dx.$$

4. Inversion Theorem: It follows easily from the one-dimensional one:

$$(u^\wedge)^\vee = u.$$

5. Convolution formula: Definition:

$$f * g(\vec{x}) = \int\int\cdots\int f(x_1 - y_1, x_2 - y_2, \cdots, x_n - y_n) g(y_1 \cdots y_n) dy_1 \cdots dy_n.$$

Formula: It follows easily from the one-dimensional one:

$$(f * g)^\wedge(\vec{\omega}) = (2\pi)^{\frac{n}{2}} \hat{f}\hat{g}.$$

Inverse: It follows easily from the one-dimensional one:

$$f * g = (2\pi)^{\frac{n}{2}} (\hat{f}\hat{g})^\vee.$$