6.2. Wave equation in $I\!R^1$

Modeling: Imagine a piece of string stretched tightly (a taut string). We measure the speed of sound $c = (\frac{T}{\rho})^{\frac{1}{2}}$, where T is tension in the string, ρ is linear mass density, both are assumed constant. Given the initial position $g(x)$, and initial velocity $h(x)$, we use a video camera to record its true motion, and a mathematical model with a computer to make a movie of the motion. We then compare the two videos. They can be made extremely close! I choose to present the mathematical solution formula only, while omit the derivation of the model, although the derivation is important and very interesting.

We consider the wave equation (vibrating string equation)

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(t,x), \quad t > 0, \quad x \in I\!R^1, \tag{1}$$

with initial conditions

$$u(0,x) = g(x), \quad \frac{\partial u}{\partial t}(0,x) = h(x). \tag{2}$$

Theorem (D'Alembert formula) A solution to (1)- (2) is

$$u(t,x) = \frac{1}{2}[g(x+ct)+g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)dyds. \tag{3}$$

Proof: We introduce the new coordinates

$$\begin{cases} \xi = x + ct, \\ \eta = x - ct. \end{cases} \tag{4}$$

Then by the chain rule,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}, \quad \frac{\partial^2 u}{\partial t^2} = c^2[\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}]$$

Thus (1) becomes

$$-4c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} = f(t,x), \quad \text{or} \quad \frac{\partial^2 u}{\partial \xi \partial \eta} = -\frac{1}{4c^2} f(\frac{\xi - \eta}{2c}, \frac{\xi + \eta}{2}).$$

Integrating twice, we find

$$u = F(\xi) + G(\eta) + \int_0^\xi \int_0^\eta -\frac{1}{4c^2} f(\frac{\xi - \eta}{2c}, \frac{\xi + \eta}{2}) d\xi d\eta. \tag{5}$$

Thus

$$u(t,x) = F(x+ct) + G(x-ct) - \frac{1}{4c^2}\int_0^{x+ct}\int_0^{x-ct} f(\frac{\xi-\eta}{2c}, \frac{\xi+\eta}{2})d\xi d\eta. \qquad (6)$$

By fitting this formula to initial data (2), we can determine F and G.

By further change of variables, we can manipulate the third term of (6) to be like formula (3).

6.3. Wave equation in $\mathbb{R}^3$.

We consider the initial value problem for the homogeneous three-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2[\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}] = 0, \quad t > 0, \qquad (7)$$

$$u(0, x_1, x_2, x_3) = 0. \qquad (8)$$

$$\frac{\partial u}{\partial t}(0, x_1, x_2, x_3) = h(x_1, x_2, x_3). \qquad (9)$$

We let

$$\hat{u}(t, w_1, w_2, w_3) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u(t, x_1, x_2, x_3)e^{i(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3)}dx_1 dx_2 dx_3,$$

$$\hat{h}(\omega_1, \omega_2, \omega_3) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} h(x_1, x_2, x_3)e^{i\omega\cdot x}dx.$$

That is, $\hat{u}$ is the Fourier transform of u in $\mathbb{R}^3$. Under this transform, equation (7) and conditions (8)-(9) become

$$\frac{\partial^2 \hat{u}}{\partial t^2} + c^2(\omega_1^2 + \omega_2^2 + \omega_3^2)\hat{u} = 0, \quad \hat{u}(0, \omega_1, \omega_2, \omega_3) = 0, \quad \frac{\partial \hat{u}}{\partial t}(0, \omega_1, \omega_2, \omega_3) = \hat{h}(\omega_1, \omega_2, \omega_3). \qquad (10)$$

The problem has a solution

$$\hat{u}(t, \omega_1, \omega_2, \omega_3) = \hat{h}(\omega_1, \omega_2, \omega_3)\frac{\sin((\omega_1^2 + \omega_2^2 + \omega_3^2)^{\frac{1}{2}}ct)}{c(\omega_1^2 + \omega_2^2 + \omega_3^2)^{\frac{1}{2}}}. \qquad (11)$$

By the inverse theorem

$$u(t, x_1, x_2, x_3) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} \hat{u}(\omega)e^{-i\vec{\omega}\cdot\vec{x}}d\vec{\omega}, \qquad (12)$$

and a series of hard calculations (see Weinberger, pp.333-335), we end up with

$$u(t, x_1, x_2, x_3) = \frac{t}{4\pi(ct)^2}\int\int_{|y-x|=ct} h(y)dS_y$$

$$= \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} h(x_1 + ct\sin\phi\cos\theta, x_2 + ct\sin\phi\sin\theta, x_3 + ct\cos\phi)\sin\phi d\phi d\theta. \quad (13)$$

Recall the spherical coordinates:

$$x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\phi$$

where θ is the angle in the (x, y) plane and ϕ is the angle away from the z-axis. This solution (13) is t times the average of h on a sphere centred at (x_1, x_2, x_3) with radius ct.

It is interesting to note that a solution to

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2(\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_3^2}) = 0, \\ u(0, x_1, x_2, x_3) = g(x_1, x_2, x_3), \\ \frac{\partial u}{\partial t}(0, x_1, x_2, x_3) = 0 \end{cases} \quad (14)$$

is simply

$$u(t, x_1, x_2, x_3) = \frac{\partial}{\partial t}[\frac{t}{4\pi(ct)^2}\int\int_{|y-x|=ct} g(y)dS_y]. \quad (15)$$

This can be seen by using the Fourier transform

$$\begin{cases} \frac{\partial^2 \hat{u}}{\partial t^2} + c^2(\omega_1^2 + \omega_2^2 + \omega_3^2)\hat{u} = 0, \\ \hat{u}(0, \omega_1, \omega_2, \omega_3) = \hat{g}(\omega_1, \omega_2, \omega_3), \\ \frac{\partial \hat{u}}{\partial t}(0, \omega_1, \omega_2, \omega_3) = 0 \end{cases} \quad (16)$$

which has a solution

$$\hat{u}(t, \omega_1, \omega_2, \omega_3) = \hat{g}\cos((\omega_1^2 + \omega_2^2 + \omega_3^2)^{\frac{1}{2}}ct). \quad (17)$$

Luckily we do not need to do any hard calculation to invert it since

$$\hat{u}(t, \omega_1, \omega_2, \omega_3) = \frac{\partial}{\partial t}[\hat{g}\frac{\sin((\omega_1^2 + \omega_2^2 + \omega_3^2)^{\frac{1}{2}}ct)}{c(\omega_1^2 + \omega_2^2 + \omega_3^2)^{\frac{1}{2}}}]$$

and thus (similar to the process from (11) to (13))

$$u(t, x_1, x_2, x_3) = \frac{\partial}{\partial t}[\hat{g}\frac{\sin((\omega_1^2 + \omega_2^2 + \omega_3^2)^{\frac{1}{2}}ct)}{c(\omega_1^2 + \omega_2^2 + \omega_3^2)^{\frac{1}{2}}}]^{\vee} = \frac{\partial}{\partial t}[\frac{t}{4\pi(ct)^2}\int\int_{|y-x|=ct} g(y)dS_y].$$

A solution to the full initial value problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2(\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_3^2}) = 0, \\ u(0, x_1, x_2, x_3) = g(x_1, x_2, x_3), \\ \frac{\partial u}{\partial t}(0, x_1, x_2, x_3) = h(x_1, x_2, x_3) \end{cases} \quad (18)$$

is the sum of the previous two solutions, which is called the *Poisson formula*:

$$
\begin{aligned}
u(t, x_1, x_2, x_3) = \quad & \tfrac{t}{4\pi(ct)^2} \int\int_{|y-x|=ct} h(y) dS_y \\
& + \tfrac{\partial}{\partial t}[\tfrac{t}{4\pi(ct)^2} \int\int_{|y-x|=ct} g(y) dS_y].
\end{aligned} \tag{19}
$$

Announcement: Final exam is on Monday Dec 16, 6:50pm–8:50pm.