C. General bounded domains, Green's function.

6.15 Poisson equation in a bounded domain, Green's function.

Given a domain $\Omega \subset I\!R^n$. Consider the problem

$$\triangle u = f(x) \quad \text{in} \quad \Omega, \tag{1}$$

$$u|_{\partial\Omega} = 0. \tag{2}$$

If Ω is not one of the special cases (interval, rectangle, disk, sphere, cylinder, half disk or quarter disk etc.), then separation of variables or transform methods do not work. The eigenvalue problem and eigenfunction expansion is a way, however there is an alternative way which reduces the work of finding a solution to (1)-(2) for arbitrary $f(x)$ to finding a solution for a single function $f(x) = \delta(x - x_0)$. This single solution is called **Green's function** and is defined as the solution to

$$\triangle G = \delta(x - x_0) \quad \text{in} \quad \Omega, \tag{3}$$

$$G|_{\partial\Omega} = 0. \tag{4}$$

Recall that our definition of fundamental solution F is

$$\triangle F = \delta(x - x_0) \quad \text{in} \quad I\!R^n$$

with the condition $F(x) \to 0$ as $|x| \to \infty$. Thus $W := G - F$ satisfies

$$\begin{cases} \triangle W &= 0 \quad \text{in} \quad \Omega, \\ W|_{\partial\Omega} &= -F. \end{cases}$$

So *Green's function $G = F + W$ is a "fundamental solution" that satisfies the zero boundary condition.*

If we multiply (3)-(4) with $f(x_0)$ and integrate in x_0, we find

$$\begin{aligned} \triangle(\textstyle\int_\Omega G(x, x_0) f(x_0) dx_0) &= \textstyle\int_\Omega \delta(x - x_0) f(x_0) dx_0 = f(x), \\ \textstyle\int_\Omega G(x, x_0) f(x_0) dx_0|_{\partial\Omega} &= 0. \end{aligned}$$

Thus

$$u = \int_\Omega G(x, y) f(y) dy$$

is a solution to

$$\begin{cases} \triangle u &= f(x) \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} &= 0 \qquad \text{on} \quad \partial\Omega. \end{cases}$$

For the simplest example, we see that the solution to

$$\frac{d^2u}{dx^2} = f(x), \quad u(0) = u(1) = 0$$

is given by

$$u(x) = \int_0^1 g(x,y)f(y)dy,$$

where

$$g(x,y) = (x-y)H(x-y) - x(1-y), \quad \text{for } 0 \leq x,\ y \leq 1.$$

where H is the Heaviside function $H(x) = 1$ for $x > 0$, and $H(x) = 0$ for $x < 0$. See Fig. 6.15.1. Simple calculation shows $g(0,y) = g(1,y) = 0$, and

$$\begin{aligned} g_x(x,y) &= H(x-y) - (1-y), \\ g_{xx} &= \delta(x-y). \end{aligned}$$

See pp. 145-146 of Keener. (Textbook of Keener identifies fundamental solutions with Green's functions.)

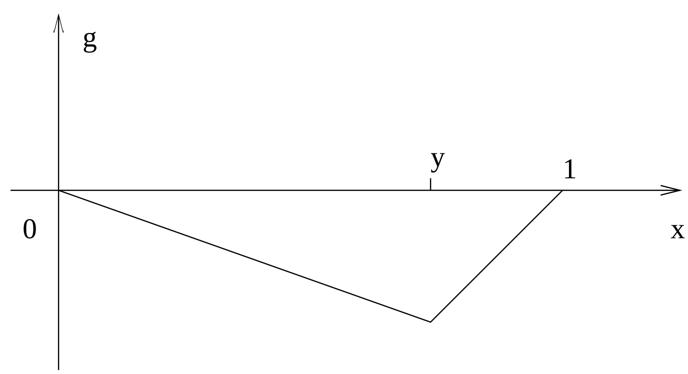

Figure 6.15.1. The Green's function for u'' = f(x), u(0) = u(1) = 0.

In general, a Green's function has no explicit formula. Still, it is helpful even when only the abstract form of existence is available. Green's functions appear in numerous situations.