

Announcement: The final exam will be cumulative. It is on Dec 16 (Monday) 6:50–8:50pm. Room: Chambers 105 for last names beginning with A – D, or Chambers 123 for last names beginning with E – Z. A mock exam was given last Monday on the web.

There will be no lecture on new materials on Dec 13, Friday. However, I will be in the classroom answering questions. It can be regarded as a review.

The student evaluation form will be done Wed.

We will cover special functions such as Bessel's functions, vibrating circular membranes, and Green's functions this week.

6.13. Explicit eigenfunctions: Orthogonal polynomials and special functions.

6.13.1. Legendre polynomials: (p.167, Keener)

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n]$$

are eigenfunctions for the differential equation

$$\begin{aligned} ((1 - x^2)u')' + \lambda u &= 0, \quad -1 < x < 1, \\ \lambda_n &= n(n + 1). \end{aligned}$$

Boundary condition is that u is bounded on $-1 \leq x \leq 1$. The function $p(x) = 1 - x^2$ vanishes at $|x| = 1$, so it is not regular. (What we covered last time in 6.12 is called a regular Sturm-Liouville eigenvalue problem. But we claim the polynomials are orthogonal and complete nonetheless.)

6.13.2. The Schrödinger equation:

$$u'' + (E - V(x))u = 0, \quad x \in \mathbb{R}^1.$$

E is eigenvalue and physicist's notation for λ . Let

$$V(x) = x^2 - 1, u = e^{-\frac{x^2}{2}} w.$$

Then

$$\begin{aligned} w'' - 2xw' + \lambda w &= 0, \\ w = H_n(x) &= (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \quad \lambda_n = 2n. \end{aligned}$$

The functions $H_n(x)$ are called the **Hermite polynomials**, which are orthogonal polynomials.

See p.167–, Keener’s for more special functions.

6.13.3. Special functions, Bessel’s differential equations.

Bessel’s differential equation of order m ($m \geq 0$) is

$$z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} + (z^2 - m^2)u = 0, \quad 0 < z < \infty.$$

We state without proof the following. It has two solutions $J_m(z)$ and $Y_m(z)$: As $z \rightarrow 0$, they satisfy

$$J_m(z) \sim \begin{cases} 1, & m = 0, \\ \frac{1}{2^m m!} z^m, & m > 0; \end{cases}$$

$$Y_m(z) \sim \begin{cases} \frac{2}{\pi} \ln z, & m = 0, \\ -\frac{2^m (m-1)!}{\pi} z^{-m}, & m > 0. \end{cases}$$

$J_m(z)$ are called Bessel functions of the *first kind*, while $Y_m(z)$ are called the *second kind*. J_m are bounded, Y_m are not bounded. Both are oscillatory and decay to zero as $z \rightarrow \infty$. See Figures 16.3.1-2. We shall let z_{mn} denote the positive zeros of $J_m(z)$, $n = 1, 2, \dots$.

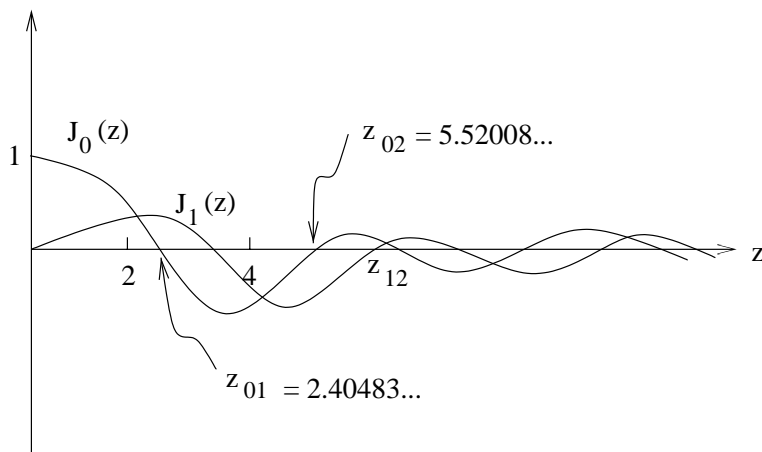


Figure 6.13.1. Bessel functions of the first kinds.

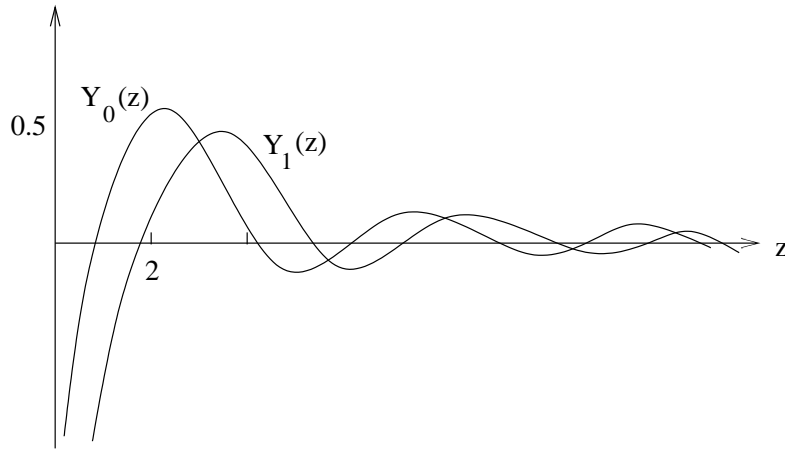


Figure 6.13.2. Bessel functions of the second kinds.

Another form of Bessel's equation: Let $z = \sqrt{\lambda r}$, where $\lambda > 0$ is a parameter, then the Bessel's equation becomes

$$r^2 \frac{d^2 u}{dr^2} + r \frac{du}{dr} + (\lambda r^2 - m^2)u = 0.$$

Or

$$(ru')' - \frac{m^2}{r}u + \lambda ru = 0, \quad 0 < r < \infty.$$

An eigenvalue problem: We propose to study the eigenvalue problem

$$\begin{cases} (ru')' - \frac{m^2}{r}u + \lambda ru = 0, & 0 < r < a, \\ |u(0)| < \infty, & u(a) = 0, \end{cases}$$

where a is any positive given number. Solution to the equation with $|u(0)| < \infty$ are

$$u(r) = cJ_m(\sqrt{\lambda r}).$$

Imposing the other boundary condition $u(a) = 0$, we have

$$\sqrt{\lambda}a = z_{mn}.$$

Thus

$$\lambda = \lambda_{mn} := \left(\frac{z_{mn}}{a}\right)^2, \quad n = 1, 2, \dots.$$

The eigenvalue problem is a singular Sturm-Liouville eigenvalue problem, we claim that orthogonality and completeness both hold:

$$\int_0^a J_m(\sqrt{\lambda_{mn}} r) J_m(\sqrt{\lambda_{mk}} r) r dr = 0, \quad n \neq k,$$

and any $\alpha(r)$, such that $\int_0^a r \alpha^2(r) dr < \infty$, has expansion

$$\alpha(r) = \sum_{n=1}^{\infty} b_n J_m(\sqrt{\lambda_{mn}} r),$$

where

$$b_n = \frac{\int_0^a \alpha(r) J_m(\sqrt{\lambda_{mn}} r) r dr}{\int_0^a J_m^2(\sqrt{\lambda_{mn}} r) r dr}.$$

This expansion is called the **Fourier-Bessel series**.