6.12. Eigenvalue problem.

6.12.1 Motivation.

We have seen that eigenvalue problems are useful in the previous sections. And they all have explicit solutions so far. Let us now look at the heat conduction equation again but with more complexity:

$$\rho c \frac{\partial u}{\partial t} = \operatorname{div}(k \nabla u). \tag{1}$$

Suppose the density ρ is now a function of $x : \rho = \rho(x)$ and $c = c(x), \quad k = k(x)$. Through separation of variables

$$u = G(t)\phi(x),$$

we end up with

$$\frac{G'}{G} = \frac{\operatorname{div}(k(x)\nabla\phi(x))}{\rho c \phi(x)} = -\lambda, \tag{2}$$

or

$$\operatorname{div}(k(x)\nabla\phi(x)) + \lambda\rho c\phi(x) = 0. \tag{3}$$

In general, there is no explicit solutions for ϕ. But we still love the idea of eigenfunction expansion. We think our elementary functions, $x^n, e^x, \ln(x), \sin(x), \arcsin(x)$, etc. are too few. We would like to establish more functions: these are called special functions and the Bessel's functions are examples.

6.12.2. Eigenvalue problem of Sturm-Liouville (p. 153 Keener).

$$\text{Equation:} \qquad \frac{d}{dx}(p(x)\frac{d\phi}{dx}) + q(x)\phi + \lambda\sigma(x)\phi = 0, \quad a < x < b, \tag{4}$$

$$\text{Boundary conditions:} \quad \begin{cases} \beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) & = 0, \\ \beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) & = 0. \end{cases} \tag{5}$$

Assumptions: $p > 0, \quad \sigma > 0, \quad p, q, \sigma$ are smooth, $|\beta_1| + |\beta_2| \neq 0, \quad |\beta_3| + |\beta_4| \neq 0$.

Conclusions:

1. All eigenvalues are real.

2. There exists an infinite number of eigenvalues:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots,$$

a. There is a smallest value λ_1,

b. There is no greatest value: $\lambda_n \to +\infty, \quad \text{as} \quad n \to \infty.$

3. Corresponding to each λ_n, there is an eigenfunction, denoted by $\phi_n(x)$ (which is unique up to a constant factor), $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.

4. The eigenfunction $\{\phi_n\}_{n=1}^{\infty}$ form a "complete" set: meaning that any L^2 integrable function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x), \qquad \text{where } a_n = \frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx},$$

in $L^2(a,b)$. Furthermore, this infinite series converges pointwise to

$$\frac{f(x+) + f(x-)}{2}$$

for $a < x < b$, provided that $f(x)$ is piecewise smooth (p. 164 Keener).

5. Weighted orthogonality:

$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)dx = 0, \quad \text{if} \quad \lambda_n \neq \lambda_m.$$

6. Any eigvenvalue can be related to its eigenfunction by the Rayleigh quotient:

$$\lambda_n = \frac{-p\phi_n\phi_n'|_a^b + \int_a^b (p\phi_n'^2 - q\phi_n^2)dx}{\int_a^b \phi_n^2 \sigma dx}.$$

Example 1:

For

$$\begin{cases} u_{xx} + \lambda u &= 0, \quad 0 < x < 1, \\ u(0) = u(1) &= 0. \end{cases}$$

We know $u_n(x) = \sin(n\pi x), \quad \lambda_n = (n\pi)^2$. But also,

$$u(x) = \lambda \int_0^1 K(x,y)u(y)dy,$$

where

$$K(x,y) = \begin{cases} y(1-x), & 0 \le y < x \le 1 \\ x(1-y), & 0 \le x < y \le 1. \end{cases}$$

Let

$$Tu := \int_0^1 K(x,y)u(y)dy.$$

Then the **eigenvalue** problem becomes the **spectrum** problem of the compact operator

$$Tu = \frac{1}{\lambda}u$$

(p. 114 Keener.)

Example 2: Heat flow in a nonuniform rod. The PDE is

$$\rho(x)c(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(k(x)\frac{\partial u}{\partial x}).$$

The initial condition is $u(0,x) = g(x)$. Let us propose the boundary condition

$$u(t,0) = 0, \quad \frac{\partial u}{\partial x}(t,L) = 0.$$

Before jumping to an eigenvalue problem, let us try to use separation of variables: $u(t,x) = G(t)\phi(x)$. Then the equation becomes

$$\frac{G'(t)}{G(t)} = \frac{(k(x)\phi'(x))'}{\rho c\phi} = -\lambda.$$

Thus we have the eigenvalue problem

$$\begin{cases} (k(x)\phi'(x))' + \lambda\rho c\phi & = 0, \\ \phi(0) & = 0, \\ \phi'(L) & = 0. \end{cases}$$

And $G(t) = be^{-\lambda t}$ where b is any constant. By Sturm-Liouville Theorem, we have λ_n and associated $\phi_n(x)$ $(n = 1,2,3,\cdots)$. So the general solutions are

$$u(t,x) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n t}\phi_n(x).$$

To find b_n, we set $t = 0$ in the above equation:

$$g(x) = \sum_{n=1}^{\infty} b_n\phi_n(x).$$

Using orthogonality with weight ρc, we find

$$\int_0^L g(x)\phi_m(x)\rho c\,dx = b_m\int_0^L \phi_m^2(x)\rho(x)c(x)\,dx.$$

So

$$b_m = \int_0^L g(x)\phi_m(x)\rho c \, dx \,/ \int_0^L \phi_m^2(x)\rho(x)c(x)\, dx. \tag{6}$$

Thus the solution is

$$u(t,x) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n t}\phi_n(x)$$

where b_n are given by (6). We do not know much details about λ_n or $\phi_n(x)$ at this point.