6.11. Wave equation in a rectangle.

6.11.1 Vibrating string with fixed ends

$$\text{PDE} \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \tag{1}$$

$$\text{Boundary Condition:} \qquad u(t,0) = u(t,L) = 0, \tag{2}$$

$$\text{Initial Condition:} \quad u(0,x) = g(x), \tag{3}$$

$$u_t(0,x) = h(x). \tag{4}$$

Propose to study the associated **eigenvalue problem**

$$\frac{\partial^2 u}{\partial x^2} = -\lambda u, \tag{5}$$

$$u(t,0) = u(t,L) = 0. \tag{6}$$

The solutions to (5)(6) are

$$\begin{aligned} u &= \phi_n(x) := \sin(\tfrac{n\pi x}{L}), \\ \lambda &= \lambda_n := (\tfrac{n\pi}{L})^2, \quad n = 1, 2, \cdots. \end{aligned}$$

Now use eigenfunction expansion:

$$\begin{aligned} u(t,x) &= \textstyle\sum_{n=1}^{\infty} C_n(t)\phi_n(x), \\ g(x) &= \textstyle\sum_{n=1}^{\infty} g_n\phi_n(x), \\ h(x) &= \textstyle\sum_{n=1}^{\infty} h_n\phi_n(x). \end{aligned}$$

Use equation (1) to obtain

$$\sum_{n=1}^{\infty} (C_n''(t) + \lambda_n c^2 C_n)\phi_n = 0.$$

Thus

$$\begin{aligned} C_n''(t) + \lambda_n c^2 C_n &= 0, \\ C_n(0) &= g_n, \\ C_n'(0) &= h_n. \end{aligned}$$

General solution formula for C_n is

$$C_n = a_n \cos(\frac{n\pi ct}{L}) + b_n \sin(\frac{n\pi ct}{L}),$$

so that

$$C_n'(t) = -a_n \frac{n\pi c}{L} \sin(\frac{n\pi ct}{L}) + b_n \frac{n\pi c}{L} \cos(\frac{n\pi ct}{L}).$$

Using initial condition, we find

$$\begin{aligned} a_n &= g_n, \\ b_n &= h_n \tfrac{L}{n\pi c}. \end{aligned}$$

Thus

$$C_n(t) = g_n \cos(\frac{n\pi ct}{L}) + h_n \frac{L}{n\pi c} \sin(\frac{n\pi ct}{L}).$$

Hence the general solution to (1)-(4) is

$$u(t,x) = \sum_{n=1}^{\infty} [g_n \cos(\frac{n\pi ct}{L}) + h_n \frac{L}{n\pi c} \sin(\frac{n\pi ct}{L})] \sin(\frac{n\pi x}{L}),$$

where

$$\begin{aligned} g_n &= \tfrac{2}{L} \int_0^L g(x) \sin(\tfrac{n\pi x}{L})\, dx, \\ h_n &= \tfrac{2}{L} \int_0^L h(x) \sin(\tfrac{n\pi x}{L})\, dx. \end{aligned}$$

Remark Traditionally, we use separation of variables $u = G(t)\phi(x)$ in (1)-(4), and then end up with the same problem of eigenvalue problem (5)-(6).

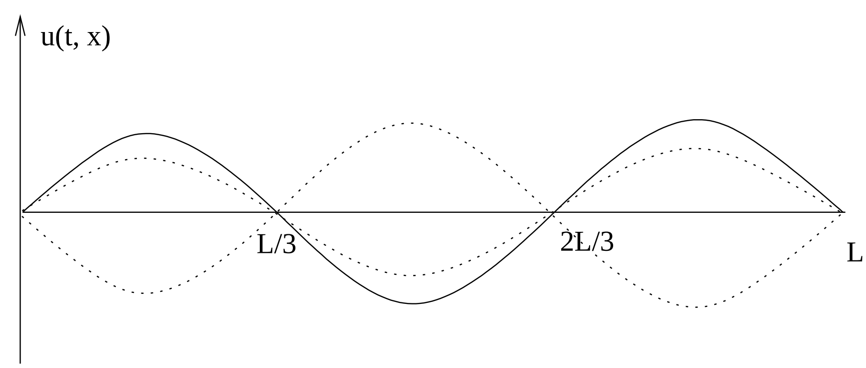

Figure 6.11.1. Normal mode at n = 3.

Property of solutions:

Let us look at the so-called **normal modes** of vibration:

$$[g_n \cos(\frac{n\pi ct}{L}) + h_n \frac{L}{n\pi c} \sin(\frac{n\pi ct}{L})] \sin(\frac{n\pi x}{L}) = A_n \sin(\frac{n\pi ct}{L} + \frac{n\pi \alpha_n}{L}) \sin(\frac{n\pi x}{L}),$$

where

$$A_n = \sqrt{g_n^2 + (\frac{h_n L}{n\pi c})^2},$$

and α_n is an angle. We see A_n is the **amplitude, temporal frequency** is $\frac{n\pi c}{L}$ and **spatial frequency** is $\frac{n\pi}{L}$. See Figure 6.11.1.

6.11.2 Vibrating rectangular membrane.

Consider

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \triangle u, \quad 0 < x < L, \quad 0 < y < H, \\ u &= 0 \qquad\qquad \text{on boundary}, \\ u(0,x,y) &= \alpha(x,y), \\ u_t(0,x,y) &= \beta(x,y). \end{aligned}$$

Eigenvalue problem:

$$\begin{aligned} \triangle u + \lambda u &= 0, \\ u &= 0 \quad \text{on boundary}. \end{aligned}$$

We know that the eigenfunctions and the eigenvalues are (from Section 6.9, Poisson equation in a rectangle, Lecture 8 of Chapter VI)

$$\begin{aligned} u &= \phi_{nm}(x,y) := \sin(\tfrac{n\pi x}{L}) \sin(\tfrac{m\pi y}{H}), \\ \lambda &= \lambda_{nm} := (\tfrac{n\pi}{L})^2 + (\tfrac{m\pi}{H})^2. \end{aligned}$$

Solutions are

$$u(t,x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [A_{nm} \cos(ct\sqrt{\lambda_{nm}}) + B_{nm} \sin(ct\sqrt{\lambda_{nm}})] \phi_{nm},$$

where

$$\begin{aligned} A_{nm} &= \tfrac{4}{LH} \int_0^L \int_0^H \alpha(x,y) \phi_{nm}(x,y) dx dy, \\ c\sqrt{\lambda_{nm}} B_{nm} &= \tfrac{4}{LH} \int_0^L \int_0^H \beta(x,y) \phi_{nm}(x,y) dx dy. \end{aligned}$$

I feel that you might have difficulty with the two-dimensional eigenvalue problem. You can either come to see me for more explanation or use the book “Elementary Applied PDE” by Haberman, or “Advanced Engineering Mathematics” by Erwin Kreyszig.