Chapter VI. Partial Differential Equations

Tentative contents

A. In infinite domains.
   6.2. wave equation in $\mathbb{R}^1$.
   6.3. wave equation in $\mathbb{R}^3$.
   6.4. wave equation in $\mathbb{R}^2$.
   6.5. Heat equation in $\mathbb{R}^n$ and $\mathbb{R}^1_+$.
   6.6. Laplace and Poisson equations in $\mathbb{R}^n$.
   6.7. Concept of fundamental solutions.

B. On rectangular domains, separation of variables.
   6.8. Laplace equation in a rectangle, Fourier series.
   6.9. Poisson equation in a rectangle.
   6.10. Heat equation in a rectangle.
   6.11. Wave equation in a rectangle.
   6.13. Explicit eigenfunctions, orthogonal polynomials, special functions, Bessel’s functions.

C. General Bounded domains, Green’s function.
   6.15. Poisson equation in general bounded domains, Green’s function

**6.1 Transport equation, method of characteristics**

We consider the simplest partial differential equation

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0, \quad t > 0, \quad x \in \mathbb{R}^1,$$

where $a$ is a constant. The general solution formula is

$$u(t, x) = g(x - at)$$

where $g(\cdot)$ is an arbitrary (smooth) function. Let $t = 0$ in (2), we see that

$$u(0, x) = g(x),$$
thus \( g(\cdot) \) is the initial condition for \( u \) and equation (1). One can let \( g \) be a Gaussian: \( g(x) = e^{-x^2} \) and plot the solution at times \( t = 1, 2, 3, \ldots, 10 \) for \( a = -2, -1, 0, 1, 2 \). We can conclude that the graph of \( u(t, x) \) is simply the graph of \( g(x) \) shifted by the amount \( at \) in the \( x \) direction.

![Figure 6.1. Transport feature (shown for positive velocity \( a \)).](image)

We consider now the transport equation in \( n \)-dimension

\[
\frac{\partial u}{\partial t} + a_1 \frac{\partial u}{\partial x_1} + a_2 \frac{\partial u}{\partial x_2} + \cdots + a_n \frac{\partial u}{\partial x_n} = 0, \quad t > 0, \quad \vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n
\]  
with initial condition

\[
u(0, \vec{x}) = g(\vec{x}).
\]  

It can be readily verified that

\[
u(t, \vec{x}) = g(\vec{x} - \vec{a}t),
\]  

Equation (4) is called a passive transport equation. We can add a source term to it and consider

\[
\left\{ \begin{array}{l}
\frac{\partial u}{\partial t} + \vec{a} \cdot \nabla u = f(t, \vec{x}), \\
u(0, \vec{x}) = g(\vec{x}).
\end{array} \right.
\]  

Let us consider the straight lines

\[
\frac{d\vec{x}}{dt} = \vec{a},
\]  
i.e.,

\[
\vec{x} = \vec{x}(t) \equiv \vec{x}_0 + \vec{a}t,
\]
which cover the whole space $\mathbb{R}^n \times \mathbb{R}$, when $\bar{x}_0$ and $t$ vary freely. These lines are called characteristic lines of equation (7). See Figure 6.2. Let us fix a $\bar{x}_0$ and consider the function $u(t, \bar{x}(t))$. We find

$$\frac{d}{dt} u(t, \bar{x}(t)) = \frac{\partial u}{\partial t} + \nabla u \cdot \frac{d}{dt} \bar{x}(t) = \frac{\partial u}{\partial t} + a \cdot \nabla u = f(t, \bar{x}(t)). \tag{10}$$

Thus we can integrate (10) to find

$$u(t, \bar{x}(t)) = u(0, \bar{x}(0)) + \int_0^t f(s, \bar{x}(s))ds = g(\bar{x}_0) + \int_0^t f(s, \bar{x}_0 + \bar{a}s) ds. \tag{11}$$

Looking at the characteristic lines the other way around, we can first fix a point $(t, \bar{x}) \in \mathbb{R}^1 \times \mathbb{R}^n$, and determine an $\bar{x}_0$ at $t = 0$ from (9), and then (11) reads as

$$u(t, \bar{x}) = g(\bar{x} - \bar{a}t) + \int_0^t f(s, \bar{x} - \bar{a}(t - s))ds.$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{characteristic_lines.png}
\caption{Characteristic lines.}
\end{figure}

**Motivation of the equation:**

Convection or transport is an important part in many partial differential equations, such as neutron transport, Boltzmann equation, fluid dynamics, etc.

The method used in (8)-(11) is called the method of characteristics. This method can be used to solve equation (7) when $\bar{a}$ is a function of $(t, \bar{x})$, or even when $\bar{a}$ is a function of $u$, making (7) a nonlinear first-order equation.