5.4. Stability of first-order linear system

Motivation: A solution needs to be stable in order to be useful in practice. The U.S. missile defense system is not yet stable.

Consider

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \vec{x}(0) = \vec{C}. \tag{1}$$

where A is an $n \times n$ matrix of constants, with n distinct eigenvalues. The solution formula is

$$\vec{x}(t) = \alpha_1 \vec{a}_1 e^{\lambda_1 t} + \alpha_2 \vec{a}_2 e^{\lambda_2 t} + \cdots + \alpha_n \vec{a}_n e^{\lambda_n t}.$$

Theorem 1. If the real parts of all the eigenvalues of the coefficient matrix A are (strictly) negative, then any solution to (1) goes to zero as $t \to +\infty$.

Theorem 2. If one or more eigenvalues of A have positive real parts, then some solutions of (1) go to infinity as $t \to +\infty$.

Proofs: They follow from the solution formula if all eigenvalues are distinct. Otherwise, solutions are like $t^m e^{\lambda t}$ which also go to zero if the real part of λ is negative, or go to infinity if the real part of λ is positive.

Now let us consider a perturbation of (1)

$$\frac{d\vec{x}}{dt} = A\vec{x} + R(t, \vec{x}). \tag{2}$$

Suppose

$$\|R(t, \vec{x})\| \le \alpha \|\vec{x}\|, \quad \text{on } \{t \ge 0, \ \|\vec{x}\| < H\}. \tag{3}$$

for some constants α and $H > 0$. Then

Theorem 3. If the real parts of all the eigenvalues of A are (strictly) negative, and (3) holds for a suitably small α, then the zero solution of (1) is asymptotically stable; *i.e.*, all solutions of (2) with small initial data go to zero as $t \to +\infty$.

Theorem 4. If one or more eigenvalues of A have positive real parts, then the zero solution is not stable, provided that (3) holds for a suitably small α.

What if one eigenvalue has zero real part and all others have negative real parts? This is called the *critical case*, and is where *bifurcation* occurs. We will discuss these issues in the next section. We provide some concrete stability examples below.

Examples 1. Consider

$$\begin{cases} \frac{dx_1}{dt} &= \lambda x_1 \\ \frac{dx_2}{dt} &= \mu x_2. \end{cases} \tag{4}$$

Suppose that $\lambda < 0, \quad \mu < 0$. Then all solutions go to zero as $t \to +\infty$. Add perturbation $R(t, x_1, x_2) : \|R(t, \vec{x})\| < \varepsilon \|\vec{x}\|$,

$$\begin{cases} \frac{dx_1}{dt} &= \lambda x_1 + R_1(t, x_1, x_2) \\ \frac{dx_2}{dt} &= \mu x_2 + R_2(t, x_1, x_2), \end{cases}$$

where $\varepsilon < \min(|\lambda|, |\mu|)$, the zero solution is stable: all solutions $x(t) \to 0$ as $t \to +\infty$.

2. For (4) again, but $\lambda < 0 < \mu$. Zero is still a solution. But it is not stable since the initially nearby solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha e^{\mu t} \end{pmatrix},$$

where α is small, grows to infinity.

3. Consider now for $\beta \neq 0$, the system

$$\begin{cases} \frac{dx_1}{dt} &= \beta x_2 \\ \frac{dx_2}{dt} &= -\beta x_1. \end{cases}$$

Differentiating the first equation and using the second equation we find

$$\frac{d^2 x_1}{dt^2} + \beta^2 x_1 = 0.$$

We can therefore find the solution formula

$$\begin{cases} x_1 &= x_1^0 \cos(\beta t) + x_2^0 \sin(\beta t) \\ x_2 &= -x_1^0 \sin(\beta t) + x_2^0 \cos(\beta t). \end{cases}$$

Introduce $\rho(t) = (x_1^2 + x_2^2)^{\frac{1}{2}}$, then $\rho(t) = ((x_1^0)^2 + (x_2^0)^2)^{\frac{1}{2}}$. See Figure 5.2 for the phase portrait of the solutions. This solution is however unstable to perturbations of the form

$$\begin{pmatrix} 0 \\ \alpha x_2 \end{pmatrix},$$

where $\alpha > 0$, because then the equation has the matrix

$$\begin{pmatrix} 0 & \beta \\ -\beta & \alpha \end{pmatrix},$$

one of whose eigenvalues has positive real part.

Notes 1. The stability of a nonzero solution $w(t)$ can be transformed to the stability of the zero solution to the equation for $v(t) \equiv u(t) - w(t)$.

2. A general nonlinear system

$$\frac{d\vec{x}}{dt} = \vec{F}(t, \vec{x})$$

may be approximateed by (2) just as a curve can be approximated by its tangent lines.

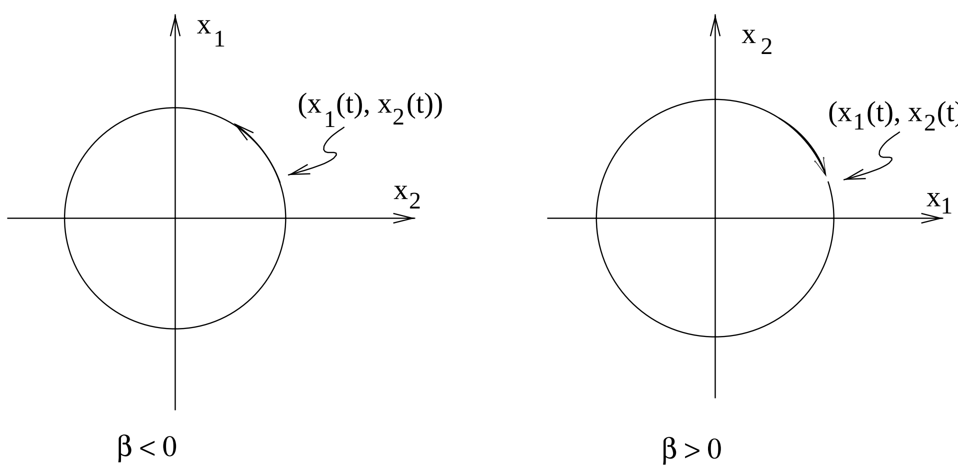

Figure 5.1. Any solution of Example 3 traces a circle.

5.5. Hopf bifurcations and example.

Motivation: Bifurcation theory is used in many life sciences, ecological systems, weather system, fluid, chaos, and turbulence.

Consider

$$\frac{d^2u}{dt^2} + (u^2 - \lambda)\frac{du}{dt} + u = 0. \tag{5}$$

It has the solution $u = 0$. Let us consider the linearized equation

$$\frac{d^2u}{dt^2} - \lambda\frac{du}{dt} + u = 0.$$

When we try solutions of the form $u = e^{\mu t}$, we find

$$\mu^2 - \lambda u + 1 = 0.$$

For $\lambda < 0$, both roots have negative real parts, so zero solution is stable. For $\lambda > 0$, both roots have positive real parts, so zero solution is unstable. At $\lambda = 0$, the roots are purely imaginary $\mu = i, -i$ and the linearized equation has periodic solutions $u = e^{it} = \cos t + i \sin t$, or $u = e^{-it} = \cos t - i \sin t$. Both the real and imaginary parts are real solutions $u(t) = \cos t$ or $\sin t$.

We write equation (5) in vector form by introduing $u_1 = u, u_2 = u'$:

$$\begin{cases} u_1' &= u_2 \\ u_2' &= (\lambda - u_1^2)u_2 - u_1. \end{cases}$$

Or

$$\vec{u}(t) = \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -u_1^2 u_2 \end{pmatrix}.$$

This nonlinear system has nonzero periodic solution near $\lambda = 0$:

$$\lambda = \frac{\varepsilon^2}{4} + O(\varepsilon^3), \quad u_1(t) = \varepsilon \cos(\omega t) + O(\varepsilon^3), \quad \omega = 1 + O(\varepsilon^3).$$

We will derive this expansion in perturbation theory next semester. For now we have a bifurcation diagram, see Figure 5.2, and we state a general bifurcation theorem called Hopf bifurcation.

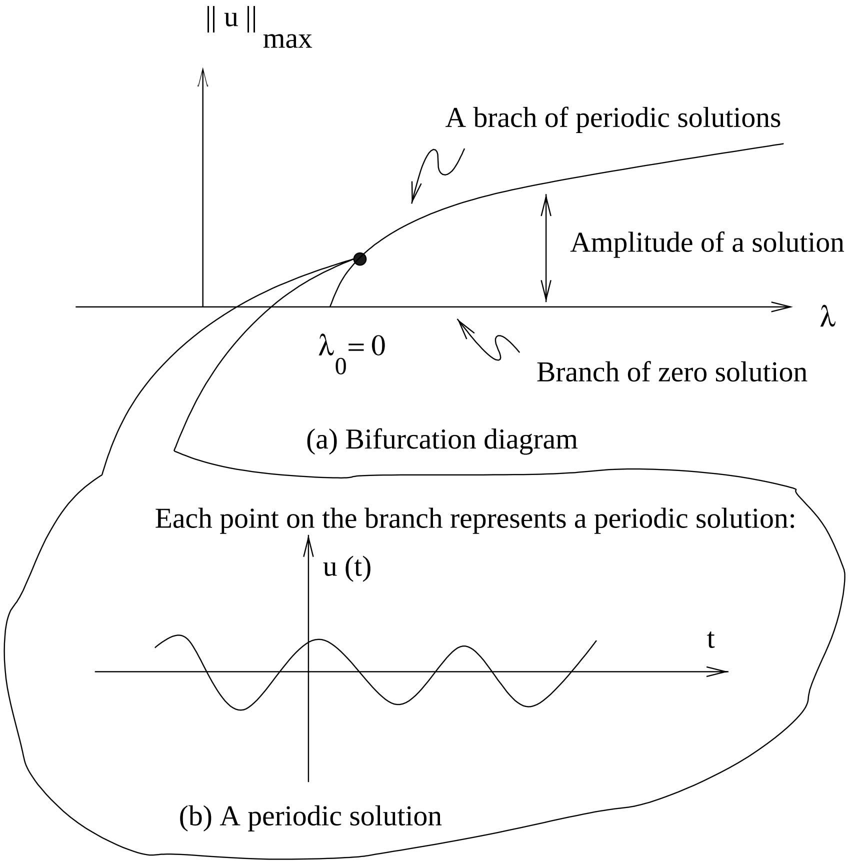

Figure 5.2. Hopf bifurcation diagram.

Theorem(Hopf Bifurcation). Suppose the $n \times n$ matrix $A(\lambda)$ has eigenvalues $\mu_j = \mu_j(\lambda), (j = 1, 2, \cdots, n)$, and that for $\lambda = \lambda_0, \quad \mu_1(\lambda_0) = i\beta, \quad \mu_2(\lambda_0) = -i\beta$ and $\mathrm{Re}\, \mu_j(\lambda_0) \neq 0$ for all $j > 2$. Suppose further that $\mathrm{Re}\,(\mu_1'(\lambda_0)) \neq 0$. Then the system of differential equations

$$\frac{du}{dt} = A(\lambda)u + f(u)$$

with $f(0) = 0, f(u)$ a smooth function of u, has a branch (continuum) of periodic solutions emanating from $u = 0, \lambda = \lambda_0$.

(The direction of bifurcation is not determinated by the Hopf Bifurcation Theorem, but must be calculated by a local power series expansion (See Keener)).

We plan to do serious perturbation theory next semester, where we can understand how a mathematician's perturbation and calculation helps locating the position of the 9th planet of the solar system.

5.6. Another bifurcation example.

See Keener p.478: Nonlinear Eigenvalue problems.

Consider the *elastica equation* (a.k.a. Euler column)

$$\begin{cases} y'' + (\lambda - \frac{1}{2}\int_0^1 (y')^2 ds) y = 0 \\ y(0) = y(1), \end{cases} \tag{6}$$

where λ is a parameter.

We see that the integral $\int_0^1 (y')^2(s)ds$ is a number. So let us introduce the number $\mu = \lambda - \frac{1}{2}\int_0^1 (y')^2(s)ds$. Then equation (6) becomes

$$y'' + \mu y = 0, \qquad y(0) = y(1), \tag{7}$$

which has solutions

$$y(x) = A\sin(n\pi x), \qquad \text{for } \mu = n^2\pi^2. \tag{8}$$

These solutions produce

$$\mu = \lambda - \frac{1}{2}\int_0^1 (y')^2(s) = \lambda - \frac{1}{2}\int_0^1 (An\pi)^2 \cos^2(n\pi x)dx = \lambda - \frac{1}{4}(An\pi)^2.$$

To satisfy (6), we need this μ to be the same as in (8); that is

$$\frac{A_n^2}{4} = \frac{\lambda}{n^2\pi^2} - 1.$$

Thus we find many branches of solutions besides the zero solution, See Figure 5.3.

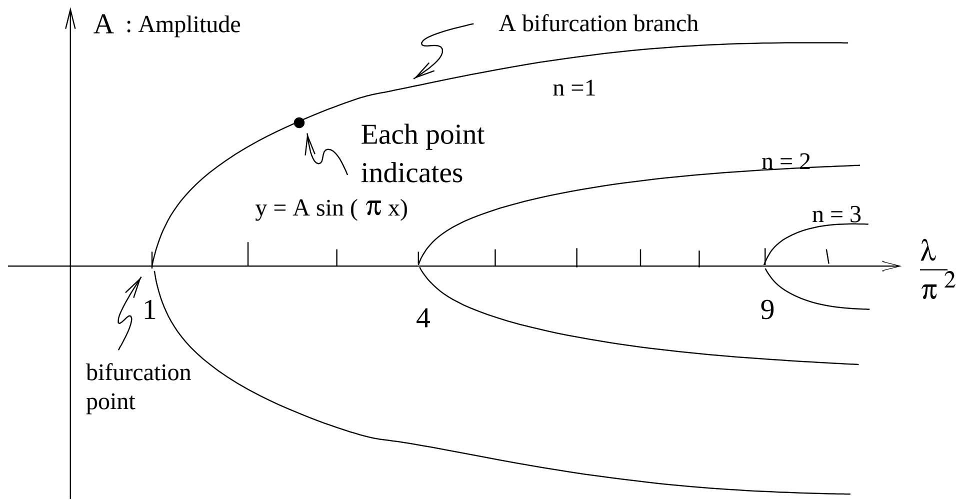

Figure 5.3. A nonlinear eigenvalue bifurcation diagram.