5.3. First-order linear systems with constant coefficients

Motivation: The planetary motion can be described by a system of equations.

Let us solve the system

$$\begin{cases} \frac{dx_1}{dt} - x_2 - x_3 = 0, \\ \frac{dx_2}{dt} - x_1 - x_3 = 0, \\ \frac{dx_3}{dt} - x_1 - x_2 = 0, \end{cases} \tag{1}$$

with initial conditions:

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}. \tag{2}$$

We try the form, motivated by the guess work for a scalar equation $x = ce^{\lambda t}$,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} ae^{\lambda t} \\ be^{\lambda t} \\ ce^{\lambda t} \end{pmatrix}. \tag{3}$$

We see

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} e^{\lambda t} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Inserting this back to (1), we have

$$\begin{cases} \lambda x_1 - x_2 - x_3 = 0, \\ \lambda x_2 - x_1 - x_3 = 0, \\ \lambda x_3 - x_1 - x_2 = 0. \end{cases}$$

This is a homogeneous linear system of three algebraic equations. We write it in matrix form

$$\begin{pmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

Remove the common factor $e^{\lambda t}$ in $[x_1, x_2, x_3]^T$ in the above equation, we find

$$\begin{pmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0. \tag{4}$$

To have a nonzero solution $[a, b, c]^T$, we need the matrix to have zero determinant:

$$\det \begin{pmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{pmatrix} = 0. \tag{5}$$

This determinant can be evaluated to be

$$\lambda^3 - 3\lambda - 2 = (\lambda + 1)^2(\lambda - 2). \tag{6}$$

The factorization is made possible by inspection and $\lambda = -1$ is a solution. Equation (5) then has three roots:

$$\lambda_1 = 2, \quad \lambda_2 = -1, \quad \lambda_3 = -1. \tag{7}$$

Using the root $\lambda_1 = 2$ in (4), we have the equation

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0. \tag{8}$$

The solutions are

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad (\alpha - \text{free}) \tag{9}$$

Thus we find the first batch of solutions

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}. \tag{10}$$

Using $\lambda_2 = -1$ in (4), we find the equation

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0. \tag{11}$$

The solutions to (11) are

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \beta \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad (\beta, \gamma \text{ free}) \tag{12}$$

We find another batch of solutions to (1):

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \beta \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + \gamma \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}. \tag{13}$$

If λ_3 is different from λ_2, we can use it to find another batch. But so far we have found plenty of solutions. We combine linearly the solutions (10) and (13) to end up with the general solution formula for (1)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + \beta \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + \gamma \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}. \tag{14}$$

We can use initial condition (2) to determine the three arbitrary constants α, β, γ (which we skip).

In general equation (1) can be written as

$$\frac{d\vec{x}}{dt} = A\vec{x} \tag{15}$$

for an $n \times n$ matrix A with constant coefficients (a_{ij}). For our previous example

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The guess work is

$$\vec{x} = \vec{a} e^{\lambda t}. \tag{16}$$

where $\vec{a}$ is a vector. Then λ needs to satisfy

$$\det(\lambda I - A) = 0 \tag{17}$$

and $\vec{a}$ satisfies

$$A\vec{a} = \lambda\vec{a}. \tag{18}$$

If the characteristic equation (17) has n roots $\lambda_1, \lambda_2, \cdots, \lambda_n$, and the eigenvalue problem (18) has n corresponding linearly independent eigenvectors $\alpha_1\vec{a}_1, \alpha_2\vec{a}_2, \cdots, \alpha_n\vec{a}_n$, then the general solution for (15) is

$$\vec{x}(t) = \alpha_1\vec{a}_1 e^{\lambda_1 t} + \alpha_2\vec{a}_2 e^{\lambda_2 t} + \cdots + \alpha_n\vec{a}_n e^{\lambda_n t}. \tag{19}$$

If, for example, $\lambda_1 = \lambda_2$, and the corresponding linearly independent eigenvectors are fewer than n, then by guess work (in addition to $\alpha_1\vec{a}_1 e^{\lambda_1 t}$),

$$\vec{x} = \alpha_2(\vec{a}_2 e^{\lambda_1 t} + \vec{a}_1 t e^{\lambda_1 t}).$$

This way a nonzero $\vec{a}_2$ can be found in the sequence of equations

$$A\vec{a}_1 = \lambda_1\vec{a}_1,$$

$$A\vec{a}_2 = \lambda_1\vec{a}_2 + \vec{a}_1.$$

And the general solution is

$$\vec{x} = \alpha_1\vec{a}_1 e^{\lambda_1 t} + \alpha_2(\vec{a}_2 + \vec{a}_1 t)e^{\lambda_1 t} + \alpha_3\vec{a}_3 e^{\lambda_3 t} \cdots + \alpha_n\vec{a}_n e^{\lambda_n t}.$$

If λ_1 is repeated more times, then higher orders of t can be used in the guess work. If, however, all eigenvalues are distinct, a theorem says that the n corresponding eigenvectors are linearly independent and (19) gives the general solution.