Chapter V. Ordinary Differential Equations

Outline:

5.1 First-order linear scalar equation.

5.2 High-order linear scalar equation with constant cofficients.

5.3 First-order linear system with constant cofficients.

5.4 Stablity of first-order linear system.

5.5 Hopf bifurcation.

We cover perturbation method next semester.

5.1. First-order linear scalar equation

Let us solve the problem

$$\frac{dy}{dt} + a(t)y = 0, \quad y(0) = C. \tag{1}$$

We find

$$\frac{y'}{y} = -a(t).$$

Integrate:

$$\begin{aligned} \ln y(t) - \ln y(0) &= -\textstyle\int_0^t a(\tau)d\tau, \\ \ln y(t) &= \ln y(0) - \textstyle\int_0^t a(\tau)d\tau, \\ y(t) &= e^{\ln y(0) - \int_0^t a(\tau)d\tau} \\ &= y(0)e^{-\int_0^t a(\tau)d\tau}. \end{aligned}$$

So the solution to (1) is

$$y(t) = Ce^{-\int_0^t a(\tau)d\tau}. \tag{2}$$

Now we consider (the first-order linear scalar equation)

$$\frac{dy}{dt} + a(t)y = f(t). \tag{3}$$

We look for a factor $m(t)$ such that

$$m(t)y' + a(t)m(t)y = [m(t)y]'.$$

From the product rule, we need $m(t)$ to satisfy

$$m'(t) = a(t)m(t)$$

From formula (2), we find an $m(t)$

$$m(t) = e^{\int_0^t a(s)ds}.$$

Equation (3) multiplied by $m(t)$ becomes

$$\frac{d}{dt}[m(t)y(t)] = m(t)f(t).$$

Thus

$$m(t)y(t) - m(0)y(0) = \int_0^t m(\tau)f(\tau)d\tau,$$

or

$$y(t) = \frac{1}{m(t)}[y(0) + \int_0^t m(\tau)f(\tau)d\tau],$$

or

$$y(t) = e^{-\int_0^t a(\tau)d\tau}[y(0) + \int_0^t e^{\int_0^\tau a(s)ds} f(\tau)d\tau]. \tag{4}$$

Examples 1. All solutions to

$$y' + 5y = 0$$

are

$$y = ce^{-5t}.$$

2. All solutions to

$$y' + t^2 y = 0$$

are

$$y = ce^{-\frac{1}{3}t^3}.$$

5.2. High-order linear scalar equations with constant coefficients

Let us solve the problem

$$\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} - 2x = 0, \tag{5}$$

$$x(0) = 0, x'(0) = 1, x''(0) = -1. \tag{6}$$

We try solutions of the form

$$x(t) = e^{\lambda t}. \tag{7}$$

Then $x' = \lambda x(t), x''(t) = \lambda^2 x(t), x'''(t) = \lambda^3 x(t)$. Thus λ needs to satisfy

$$\lambda^3 x(t) + \lambda^2 x(t) - 2x(t) = 0.$$

But $x(t) = e^{\lambda t}$ is not zero, so we need λ to satisfy

$$\lambda^3 + \lambda^2 - 2 = 0.$$

We find $\lambda_1 = 1, \quad \lambda_2 = -1+i, \quad \lambda_3 = -1-i$. So we have solutions $e^t, e^{-t+it}, e^{-t-it}$. The two complex solutions can be added or subtracted one from the other to produce two real solutions, so we have three real solutions $e^t, e^{-t}\cos t, e^{-t}\sin t$, since the equation is linear. Also, any linear combination of the three solutions are solutions. Thus, we have

$$x(t) = c_1 e^{-t} + c_2 e^t \cos t + c_3 e^{-t} \sin t \tag{8}$$

as the general solution formula for (5). One can use the three initial conditions (6) to determine the three coefficients c_1, c_2, c_3 in (8) which we omit here.

In general the $n-$th-order linear scalar equation

$$a_n x^{(n)}(t) + a_{n-1} x^{(n-1)}(t) + ... + a_1 x' + a_0 x = 0$$

with constant coefficients $(a_n, a_{n-1}, \cdots, a_0)$ without forcing(right-hand side=0) can be solved by the guess work (7). More precisely, from the algebraic equation,

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0,$$

we can find n roots. Suppose it has n distinct roots $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then the solution for the ODE is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_n e^{\lambda_n t}.$$

If λ_1 and λ_2 are a pair of conjugate complex roots, say,

$$\lambda_1 = a + bi, \quad \lambda_2 = a - bi,$$

then we can replace the part $c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ by real solution of the form

$$\alpha e^{at} \cos(bt) + \beta e^{at} \sin(bt).$$

If λ_1 is repeated, say $\lambda_1 = \lambda_2$, then $c_2 e^{\lambda_2 t}$ is a multiple of the first solution $c_1 e^{\lambda_1 t}$. In this case, we replace the guess work $e^{\lambda t}$ by $te^{\lambda t}$, and the solution $c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ is replaced by $c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$. If λ_1 is repeaded m times, then the solution part

$$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_m e^{\lambda_m t}$$

is replaced by

$$c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t} + \cdots + c_m t^{m-1} e^{\lambda_1 t}.$$

Example 3. Solve

$$\frac{d^2 x}{dt^2} - 2\frac{dx}{dt} + x = 0.$$

Solution. Try $x = e^{\lambda t}$, we find

$$\lambda^2 - 2\lambda + 1 = 0.$$

So

$$\lambda_1 = \lambda_2 = 1.$$

So the solutions are

$$x(t) = c_1 e^t + c_2 t e^t.$$