4.2. Laplace transform

Definition. For any $f(t) \in L^1([0,\infty))$, the function

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\, dt$$

for $s \geq 0$, is called the Laplace transform of $f(t)$.

Laplace transform can be obtained from Fourier transform through a certain specilization. Laplace transform is very convenient to use for certain differential equations or with certain boundary condition. But in general the two linear transforms are basically the same.

Property a. $\mathcal{L}[\frac{df}{dt}](s) = s\mathcal{L}[f] - f(0)$.

Proof. We have the calculation

$$\begin{aligned} \mathcal{L}[f'] &= \textstyle\int_0^\infty e^{-st} f'(t)\, dt \\ &= e^{-st} f(t)|_0^\infty - \textstyle\int_0^\infty e^{-st}(-s) f(t)\, dt \\ &= -f(0) + s \textstyle\int_0^\infty e^{-st} f(t)\, dt \\ &= s\mathcal{L}[f] - f(0). \end{aligned}$$

We list other properties below without proof. We let $F(s)$ denote $\mathcal{L}[f](s)$; i.e., we use the capital letter to denote the Laplace transform of a lower-case letter function.

Properties

b. $\mathcal{L}[1] = \frac{1}{s}$,

c. $\mathcal{L}[e^{at}] = \frac{1}{s-a}$,

d. $\mathcal{L}[f''] = s^2 F(s) - sf(0) - \frac{df}{dt}(0)$,

e. $\mathcal{L}[-tf(t)] = \frac{dF}{ds}$,

f. $\mathcal{L}[\int_0^t f(t-\bar{t}) g(\bar{t})\, d\bar{t}] = F(s)G(s)$,

g. $\mathcal{L}[\delta(t-b)] = e^{-bs}, \quad (b \geq 0)$

h. $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad (n > -1)$,

i. $\mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}}, \quad (n > -1)$.

We note that Laplace transform transforms differentiation to a multiplication by the independent variable s, and the multiplication by t to the differentiation with respect to s. It also transforms the convolution

$$\int_0^t f(t-\bar{t}) g(\bar{t})\, d\bar{t}$$

to the product of the transforms of f and g. Note that Laplace transform does not need $f(t)$ to be defined in $t < 0$. Even if $f(t)$ is defined in $t < 0$, its value there does not affect the transform. Therefore we adopt the convention that all relevant functions for the Laplace transform are defined by zero in $t < 0$. Under this convention we find that

$$\int_0^t f(t-\bar{t})g(\bar{t})\,d\bar{t} = \int_{-\infty}^{\infty} f(t-\bar{t})g(\bar{t})\,d\bar{t} = f * g$$

is indeed the the convolution defined in the previous section.

We also note that Laplace transforms exist for functions or functionals that are not in $L^1([0,\infty))$, e.g., the functional $\delta(t-b)$, and t^2.

We mention another example. Consider the Heaviside function

$$H(t-b) = \begin{cases} 0, & t < b, \\ 1, & t \geq b. \end{cases}$$

For $b > 0$, we find that

Property j. $\mathcal{L}[H(t-b)](s) = \int_0^\infty H(t-b)e^{-st}\,dt = \int_b^\infty e^{-st}\,dt = \frac{e^{-bs}}{s}$.

Similar to the Fourier transform, there exists the inverse transform for the Laplace transform. But its use is inconvenient. The best way has been to use the above list of properties $a - j$ for the inversion. If $F(s)$ is the Laplace transform of $f(t)$, then we call $f(t)$ the inverse of $F(s)$. For example,

$$\mathcal{L}^{-1}[\frac{1}{s}] = 1, \quad (t > 0).$$

Example 1. Solve the initial value problem

$$\begin{aligned} u'' + 2u' + 2u &= 0, \quad t > 0 \\ u(0) &= 1, \\ u'(0) &= 2, \end{aligned}$$

Physical background. This equation can be regarded as the motion of a particle with mass $m = 1$, attached to a spring with Hooke's spring constant $k = 2$, and wind drag force proportional to the velocity u'. Newton's second law says $F = ma$ (Force = mass $\times$ acceleration). Here $ma = u''$ where $u(t)$ represents the displacement of the particle from the equilibrium. The spring force is $-2u$, the wind drag force is $-2u'$, where u' is velocity. So $u'' = -2u' - 2u$ is Newton's law.

Solution: Let $U(s) = \mathcal{L}[u]$. Then we use the properties $a - j$ to find

$$\begin{aligned} \mathcal{L}[u'] &= sU - 1, \\ \mathcal{L}[u''] &= s^2U - s - 2. \end{aligned}$$

So

$$\begin{aligned} s^2U - s - 2 + 2(sU - 1) + 2U &= 0, \\ (s^2 + 2s + 2)U &= s + 4, \\ U &= \tfrac{s+4}{s^2+2s+2}. \end{aligned}$$

We notice $s^2 + 2s + 2 = [s + (1 - i)][s + (1 + i)]$. By *partial fractions* (see later), we have

$$\frac{s+4}{s^2+2s+2} = \frac{\alpha}{s+1-i} + \frac{\beta}{s+1+i},$$

where

$$\alpha = \frac{1}{2}(1 - 3i), \qquad \beta = \frac{1}{2}(1 + 3i).$$

Then by linearity of the transform, we have

$$u = \mathcal{L}^{-1}(U(s)) = \alpha\mathcal{L}^{-1}[\frac{1}{s - (-1 + i)}] + \beta\mathcal{L}^{-1}[\frac{1}{s - (-1 - i)}].$$

Using property c for $a = -1 + i$ and then $a = -1 - i$, we have

$$\begin{aligned} u &= \alpha e^{(-1+i)t} + \beta e^{(-1-i)t} \\ &= \tfrac{1}{2}(1 - 3i)e^{-t}(\cos t + i \sin t) + \tfrac{1}{2}(1 + 3i)e^{-t}(\cos t - i \sin t) \\ &= e^{-t}(\cos t + 3 \sin t). \end{aligned}$$

Partial fractions. We show here how we express a complicated fraction into a sum of simple fractions for which we can invert the Laplace transform. We make a guess of the sum:

$$\frac{s+4}{s^2+2s+2} = \frac{\alpha}{s+1-i} + \frac{\beta}{s+1+i}$$

where α and β are to be determined numbers. Then we multiply the two sides of the equation with $s^2 + 2s + 2$ to find

$$s + 4 = \alpha s + \beta s + \alpha(1 + i) + \beta(1 - i).$$

We rearrange terms to find

$$(\alpha + \beta - 1)s + \alpha(1 + i) + \beta(1 - i) - 4 = 0.$$

This equation has to be true for all s, so we have

$$\begin{aligned} \alpha+\beta &= 1, \\ \alpha(1+i)+\beta(1-i) &= 4. \end{aligned}$$

This system of algebraic equations can be solved easily:

$$\begin{aligned} 1+i(\alpha-\beta) &= 4, \\ \alpha-\beta &= -3i, \\ \alpha &= \tfrac{1}{2}(1-3i), \\ \beta &= \tfrac{1}{2}(1+3i). \end{aligned}$$

This finishes the partial fraction used in our example 1. For general ways of partial fractions, see our text book.

Example 2. Solve the initial value problem of the ordinary differential equation (ODE)

$$\begin{cases} u''+tu'+u=0, \quad t>0, \\ u(0)=1, \\ u'(0)=0. \end{cases}$$

Solution: Let $U(s)=\mathcal{L}[u]$. Then

$$\begin{aligned} \mathcal{L}[u'] &= sU-1, \\ \mathcal{L}[u''] &= s^2U-s, \\ \mathcal{L}[tu'] &= -\tfrac{d}{ds}[sU-1] \\ &= -sU'-U(s). \end{aligned}$$

Then the ODE in question becomes

$$-sU'+s^2U-s=0,$$

or

$$U'-sU=-1.$$

How to solve this new ODE? We multiply it with $e^{-\frac{s^2}{2}}$, so

$$(e^{-\frac{s^2}{2}}U)'=-e^{-\frac{s^2}{2}}.$$

We integrate it in s from s to ∞ and use the condition $U(s)\to 0$ as $s\to\infty$:

$$0-e^{-\frac{s^2}{2}}U(s)=-\int_s^{\infty} e^{-\frac{\sigma^2}{2}}\,d\sigma.$$

Or we have

$$U(s) = e^{\frac{s^2}{2}} \int_s^{\infty} e^{-\frac{\sigma^2}{2}} d\sigma.$$

Instead of evaluating this integral and then inverting it, we use a special trick. We introduce the new variable $t = \sigma - s$. The integral becomes

$$\begin{aligned} U(s) &= \int_s^{\infty} e^{-\frac{\sigma^2 - s^2}{2}} d\sigma, \\ U(s) &= \int_0^{\infty} e^{-st} e^{-\frac{t^2}{2}}\, dt. \end{aligned}$$

Hence $U(s)$ is the Laplace transform of $e^{-\frac{t^2}{2}}$, and so

$$u(t) = e^{-\frac{t^2}{2}}$$

is the solution. (Reference book: Weinberger).

For more on Laplace and Fourier transforms, see

1. Richard Haberman: *Elementary Applied Partial Differential Equations*, 2nd or later edtions, Prentice Hall, 1987.

2. H. F. Weinberger: *A First Course in Partial Differential Equations*, John Wiley & Sons, 1965.