4.1. Fourier integral transform (Continued)

Definition. For any $f(x) \in L^1(I\!R^1)$, the function

$$f^{\vee}(\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\mu x} f(x) dx$$

is called the inverse Fourier transform of $f(x)$.

Note: I do not know how to type set the wedge so that it has the same size as the hat. Right now the wedge looks bigger than the hat.

Note: In terms of real functions, we have

$$f^{\vee}(\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(\mu x) f(x) dx - i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(\mu x) f(x) dx.$$

If $f(x)$ is an even function: $f(-x) = f(x)$, then the inverse Fourier transform is real and equals to the Fourier transform.

The inverse Fourier transform has similar properties as the Fourier transform:

$$\begin{aligned}
[f'(x)]^{\vee}(\mu) &= i\mu f^{\vee}(\mu) \\
\left(e^{-\beta x^2}\right)^{\vee} &= \tfrac{1}{\sqrt{2\beta}} e^{-\frac{\mu^2}{4\beta}} \\
\left(e^{-\frac{x^2}{2}}\right)^{\vee} &= e^{-\frac{\mu^2}{2}} \\
[\delta(x)]^{\vee}(\mu) &= \tfrac{1}{\sqrt{2\pi}} \\
1^{\vee}(\mu) &= \sqrt{2\pi}\delta(\mu) \\
\left(\tfrac{1}{\sqrt{2\pi}}\right)^{\vee} &= \delta(\mu).
\end{aligned}$$

We then see that there holds $(\delta(x)^{\vee})^{\vee} = \delta(x)$. A more appropriate way to write this is $\delta(x)^{\wedge\vee} = \delta(x)$. In general we have this Fourier inversion theorem.

Theorem (Fourier Inversion) For any $f(x) \in L^1(I\!R^1)$ such that $\hat{f}(\mu) \in L^1(I\!R^1)$, there holds

$$\hat{f}^{\vee}(x) = f(x).$$

Proof. We shall utilize the transform

$$1^{\wedge}(\mu) = \sqrt{2\pi}\delta(\mu)$$

in the form

$$1^{\wedge}(y-x) = \frac{1}{\sqrt{2\pi}} \int_{I\!R^1} 1 \cdot e^{i\mu(y-x)}\, d\mu = \sqrt{2\pi}\delta(y-x).$$

Thus we calculate

$$\begin{aligned}(\hat{f})^{\vee}(x) &= \tfrac{1}{\sqrt{2\pi}}\int_{I\!R^1}[\tfrac{1}{\sqrt{2\pi}}\int_{I\!R^1} f(y)e^{i\mu y}\,dy]e^{-i\mu x}d\mu \\ &\qquad \text{then change order of integration} \\ &= \tfrac{1}{\sqrt{2\pi}}\int_{I\!R^1} f(y)\left(\tfrac{1}{\sqrt{2\pi}}\int_{I\!R^1} e^{i\mu(y-x)}d\mu\right)\,dy \\ &\qquad \text{use the transform of 1} \\ &= \tfrac{1}{\sqrt{2\pi}}\int_{I\!R^1} f(y)\sqrt{2\pi}\delta(y-x)\,dy \\ &= f(x) \qquad\qquad \text{woa!}\end{aligned}$$

You may note that we do not have to use the factor $\frac{1}{\sqrt{2\pi}}$ in front of the Fourier transform. In fact many books use different factors. We choose to use this factor so that the inverse transform has the same factor as the (forward) Fourier transform, the inversion theorem holds, and the following Parseval (or Plancherel) identity holds without extra factors.

Theorem. (Parseval (or Plancherel) identity) Both Fourier and inverse Fourier transforms are bounded linear transforms that preserve the norm and distance in $L^2(I\!R^1)$:

$$\|\hat{f}\|_{L^2} = \|f^{\vee}\|_{L^2} = \|f\|_{L^2}.$$

Proof. (Not covered in class) We calculate

$$\begin{aligned}\|\hat{f}\|_{L^2}^2 &= \textstyle\int_{I\!R^1} \hat{f}(\mu)\overline{\hat{f}(\mu)}d\mu \\ &\qquad \text{overline means complex conjugate, this is by definition} \\ &= \tfrac{1}{2\pi}\textstyle\int_{I\!R^1}\int_{I\!R^1} e^{i\mu x}f(x)\,dx\,\int_{I\!R^1} e^{-i\mu y}f(y)\,dy\,d\mu \\ &= \tfrac{1}{\sqrt{2\pi}}\textstyle\int_{I\!R^1}\int_{I\!R^1}\left(\tfrac{1}{\sqrt{2\pi}}\int_{I\!R^1} e^{i\mu(x-y)}d\mu\right) f(x)f(y)\,dx\,dy \\ &= \tfrac{1}{\sqrt{2\pi}}\textstyle\int_{I\!R^1}\int_{I\!R^1}\sqrt{2\pi}\delta(y-x)f(x)f(y)\,dx\,dy \\ &= \textstyle\int_{I\!R^1} f(x)f(x)\,dx = \|f\|_{L^2}^2.\end{aligned}$$

Definition (Convolution) Let $f(x)$ and $g(x)$ be in $L^1(I\!R^1)$. Then the function

$$(f*g)(x) = \int_{I\!R^1} f(x-y)g(y)\,dy$$

is called the convolution of the two functions f and g.

We note that we can use a change of variables to find

$$(f*g)(x) = \int_{I\!R^1} f(y)g(x-y)\,dy.$$

Property b. There holds

$$(f * g)^{\hat{}}(\mu) = \sqrt{2\pi} f^{\hat{}}(\mu) \cdot g^{\hat{}}(\mu)$$

and

$$(f * g)^{\vee}(\mu) = \sqrt{2\pi} f^{\vee}(\mu) \cdot g^{\vee}(\mu)$$

Proof. We calculate

$$\begin{aligned}
(f * g)^{\hat{}}(\mu) &= \tfrac{1}{\sqrt{2\pi}} \textstyle\int_{I\!R^1} (\int_{I\!R^1} f(x-y) g(y)\, dy) e^{i\mu x}\, dx \\
&\qquad \text{change order of integration} \\
&= \textstyle\int_{I\!R^1} \left[\tfrac{1}{\sqrt{2\pi}} \int_{I\!R^1} f(x-y) e^{i\mu(x-y)}\, dx \right] g(y) e^{i\mu y}\, dy \\
&\qquad \text{change of variable } x - y = z \\
&= \textstyle\int_{I\!R^1} \left[\tfrac{1}{\sqrt{2\pi}} \int_{I\!R^1} f(z) e^{i\mu z}\, dz \right] g(y) e^{i\mu y}\, dy \\
&= \sqrt{2\pi} f^{\hat{}}(\mu) \cdot g^{\hat{}}(\mu).
\end{aligned}$$

We can prove the second formula similarly.

Property c. There holds

$$(fg)^{\hat{}} = \frac{1}{\sqrt{2\pi}} f^{\hat{}} * g^{\hat{}} \quad \text{and} \quad (fg)^{\vee} = \frac{1}{\sqrt{2\pi}} f^{\vee} * g^{\vee}.$$

Proof. By applying the Fourier transform to the second formula of property b, we have

$$(f * g)(x) = \sqrt{2\pi} (f^{\vee}(\mu) \cdot g^{\vee}(\mu))^{\hat{}}.$$

Call $F = f^{\vee}, G = g^{\vee}$, then $F^{\hat{}} = f, G^{\hat{}} = g$. So we have

$$(F^{\hat{}} * G^{\hat{}})(x) = \sqrt{2\pi} (F \cdot G)^{\hat{}}.$$

Dividing by $\sqrt{2\pi}$, and realizing that F and G are arbirary, so we can change them to f and g to find

$$(f \cdot g)^{\hat{}} = \frac{1}{\sqrt{2\pi}} (f^{\hat{}} * g^{\hat{}}).$$

The proof for the other formula is similar.

Examples 1. Solve the heat conduction problem in an infinite slab:

$$\begin{aligned}
\tfrac{\partial u}{\partial t} - \tfrac{\partial^2 u}{\partial x^2} &= 0 \qquad x \in I\!R^1,\ t > 0 \\
u(x, 0) &= f(x).
\end{aligned}$$

Solution. We use the Fourier transform and notation

$$u\hat{}(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} u(x,t)dx.$$

Since t is independent of x, we have

$$\partial_t u\hat{}(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \partial_t u(x,t)dx.$$

Also we have

$$(-i\omega)^2 u\hat{}(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \partial_x^2 u(x,t)dx.$$

We transform the initial data:

$$u\hat{}(\omega,0) = f\hat{}(\omega).$$

Now we transform the equation: We multiply the equation with the factor $e^{i\omega x}$, integrate the equation in x over $I\!R^1$, we have

$$\partial_t(u\hat{}) + \omega^2 u\hat{} = 0.$$

We can find the solution to the ordinary differential equation (see later):

$$u\hat{}(\omega,t) = f\hat{}(\omega)e^{-t\omega^2}.$$

Let g be such a function:

$$g\hat{} = e^{-t\omega^2}.$$

We know that

$$g(x) = (e^{-t\omega^2})^{\vee} = \frac{1}{\sqrt{2t}} e^{-\frac{x^2}{4t}}.$$

Thus we have

$$u\hat{}(\omega,t) = f\hat{}g\hat{}.$$

Hence

$$u(x,t) = (u\hat{}(\omega,t))^{\vee} = (f\hat{}g\hat{})^{\vee} = \frac{1}{\sqrt{2\pi}} f * g = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2t}} \int_{-\infty}^{\infty} f(y)e^{-\frac{(x-y)^2}{4t}}\, dy.$$

Thus

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(y)e^{-\frac{(x-y)^2}{4t}}\, dy.$$

2. Solve the ordinary differential equation

$$-u'' + a^2 u = f(x) \in L^2(\mathbb{R}^1).$$

Solution. We have

$$-u'' e^{i\mu x} + a^2 u e^{i\mu x} = f(x) e^{i\mu x}.$$

Integrating in x, we find

$$-(-i\mu)^2 u\hat{} + a^2 u\hat{} = f\hat{}.$$

Or

$$(\mu^2 + a^2) u\hat{} = f\hat{}.$$

Or simply

$$u\hat{} = \frac{1}{\mu^2 + a^2} f\hat{} = g\hat{} f\hat{}$$

where

$$g\hat{} = \frac{1}{\mu^2 + a^2}.$$

We will show later that

$$g = \sqrt{2\pi} \frac{e^{-a|x|}}{2a}.$$

Then we have the solution formula:

$$u(x) = \frac{1}{\sqrt{2\pi}} f * g = \frac{1}{2a} \int_{-\infty}^{\infty} f(x-y) e^{-a|y|}\, dy.$$

End of solution.

We find g here. First we can use integration by parts twice to find that

$$I = \int_0^\infty \cos(\mu x) e^{-ax}\, dx = \frac{1}{a} - \frac{\mu^2}{a^2} I.$$

We solve for I:

$$I = \frac{a}{a^2 + \mu^2}.$$

Now we verify

$$\begin{aligned} g\hat{} &= \textstyle\int_{\mathbb{R}^1} e^{i\mu x} \frac{e^{-a|x|}}{2a}\, dx \\ &= \textstyle\int_{\mathbb{R}^1} \cos(\mu x) \frac{e^{-a|x|}}{2a}\, dx + i \cdot 0(\text{ odd integrand }) \\ &= \textstyle\frac{1}{a} \int_0^\infty \cos(\mu x) e^{-ax}\, dx \\ &\equiv \textstyle\frac{1}{a} I = \frac{1}{a^2+\mu^2} = g\hat{}. \end{aligned}$$

Here we solve the ordinary differential equation

$$\partial_t(u\hat{}) + \omega^2 u\hat{} = 0.$$

For convenience, we call $y(t) = u\hat{}$. Then the equation becomes

$$\partial_t y + \omega^2 y = 0.$$

Dividing the equation by y, we have

$$\frac{\partial_t y}{y} = -\omega^2.$$

Using the chain rule of differentiation $[F(g(x))]' = F'(g(x))g'(x)$, we have

$$\partial_t \ln y = -\omega^2.$$

Integrating the equation in t from $t = 0$, we have

$$\ln y(t) - \ln y(0) = -\omega^2 t.$$

Taking the exponential on both sides, we have

$$y(t) = y(0)e^{-\omega^2 t}.$$

Recall that $y(0) = f\hat{}(\omega)$. So we have finally

$$u\hat{}(\omega, t) = f\hat{}(\omega)e^{-t\omega^2}.$$