Chapter IV. Fourier and Laplace Transforms

Course contents outline:

4.1. Fourier integral transform, properties, examples;

4.2. Laplace transform, properties, examples.

4.1. Fourier integral transform

No appropriate part from the text by Keener.

Let $I\!R^1 = (-\infty, \infty)$.

Definition. For any $f(x) \in L^1(I\!R^1)$, the function

$$\hat{f}(\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\mu x} f(x)dx$$

is called the Fourier (integral) transform of $f(x)$.

Note: In terms of real functions, we have

$$\hat{f}(\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(\mu x) f(x)dx + i\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(\mu x) f(x)dx.$$

Property a. $\left(\frac{df}{dx}\right)\hat{}\,(\mu) = -i\mu\hat{f}(\mu)$.

Proof. By definition, we have

$$\begin{aligned}
\left(\tfrac{df}{dx}\right)\hat{}\,(\mu) \quad &= \tfrac{1}{\sqrt{2\pi}} \textstyle\int_{-\infty}^{\infty} \tfrac{df}{dx}(x) e^{i\mu x} dx \\
&\quad \text{use integration by parts} \\
&= \tfrac{1}{\sqrt{2\pi}} \left[f(x)e^{i\mu x}|_{-\infty}^{\infty} - \textstyle\int_{-\infty}^{\infty} (e^{i\mu x})' f(x)dx \right] \\
&\quad \text{use } f \to 0 \text{ as } |x| \to \infty \\
&= \tfrac{1}{\sqrt{2\pi}}(-i\mu) \textstyle\int_{-\infty}^{\infty} f(x)e^{i\mu x}dx \\
&= -i\mu\hat{f}(\mu). \text{ — END of PROOF.}
\end{aligned}$$

We can feel that it is not going to be an easy job to calculate Fourier transforms for a given function $f(x)$. We show a few examples.

Examples 1. Consider $f(x) = e^{-\beta x^2}$ $(\beta > 0)$. We draw it in Figure 4.1 for two different β. The graphs always pass through the point $(x, y) = (0, 1)$ and have the shape of a bell.

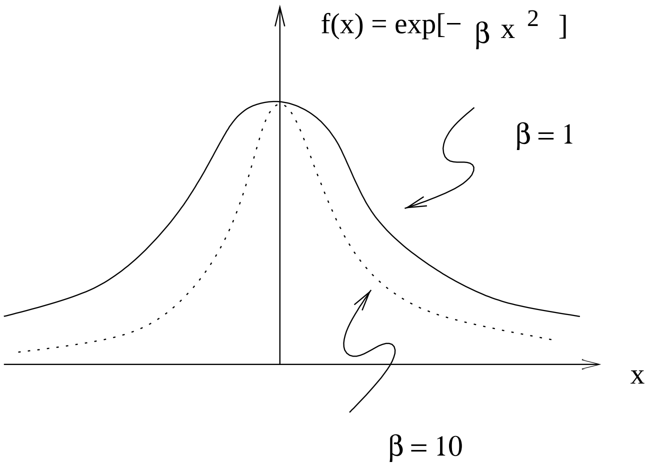

Figure 4.1. Bell shaped functions.

Then

$$\hat{f}(\mu) = \frac{1}{\sqrt{2\beta}} e^{-\frac{\mu^2}{4\beta}}.$$

Proof. Step 1. We assume that you know the integral

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \tag{1}$$

Then, by a simple change of variables we find

$$\int_{-\infty}^{\infty} e^{-\beta x^2} dx = \frac{1}{\sqrt{\beta}} \int_{-\infty}^{\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{\sqrt{\beta}}.$$

By using contour integral (see our lecture notes Chapter II, Theorem 2.2), see Figure 4.2, we can show that

$$\int_{\Gamma} e^{-\beta z^2} dz = \frac{\sqrt{\pi}}{\sqrt{\beta}}$$

where Γ is any straight line $z = x + bi$ for any real constant b where $x \in I\!R^1$. You see that the case $b = 0$ is the real $x-$axis. Thus

$$\int_{-\infty}^{\infty} e^{-\beta(x - \frac{\mu i}{2\beta})^2} dx = \frac{\sqrt{\pi}}{\sqrt{\beta}}$$

for all μ.

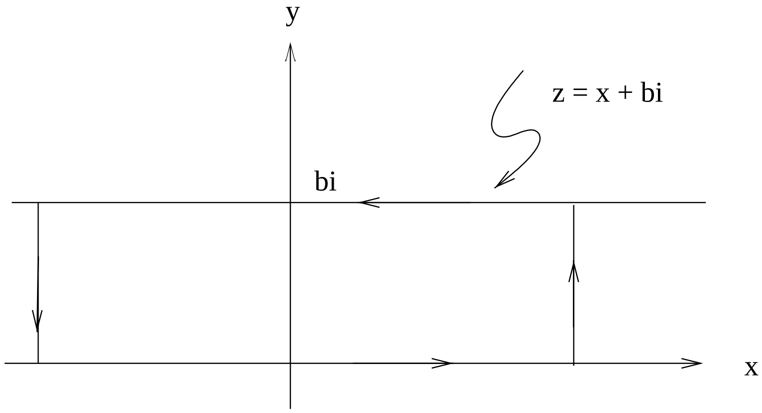

Figure 4.2. The contour.

Step 2. We have

$$\begin{aligned}\hat{f}(\mu) &= \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\beta x^2+i\mu x+\frac{\mu^2}{4\beta}-\frac{\mu^2}{4\beta}}\,dx\\ &= \tfrac{e^{-\frac{\mu^2}{4\beta}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\beta(x-\frac{\mu i}{2\beta})^2}\,dx\\ &= \tfrac{e^{-\frac{\mu^2}{4\beta}}}{\sqrt{2\pi}}\cdot\tfrac{\sqrt{\pi}}{\sqrt{\beta}} = \tfrac{1}{\sqrt{2\beta}}e^{-\frac{\mu^2}{4\beta}}.\end{aligned}$$

If you do not know the integral in (1), then here is a cute proof.

$$\begin{aligned}I^2 &= \int_{-\infty}^{\infty} e^{-x^2}dx\cdot\int_{-\infty}^{\infty} e^{-y^2}dy\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy\\ &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta\\ &= 2\pi\int_0^{\infty} e^{-r^2}\,r\,dr\\ &= \pi\int_0^{\infty} e^{-w}dw = \pi.\end{aligned}$$

This completes the proof.

2. For $\beta = \frac{1}{2}$, we have

$$\left(e^{-\frac{x^2}{2}}\right)\hat{} = e^{-\frac{\mu^2}{2}}.$$

That is, the Fourier transform of the Gaussian $e^{-\frac{x^2}{2}}$ is itself.

3. From Example 1 we can have

$$\left(\frac{\sqrt{\beta}}{\sqrt{\pi}}e^{-\beta x^2}\right)\hat{} = \frac{1}{\sqrt{2\pi}}e^{-\frac{\mu^2}{4\beta}}.$$

Let

$$g_\beta(x) = \frac{\sqrt{\beta}}{\sqrt{\pi}} e^{-\beta x^2},$$

we see that

$$g_\beta(x) \to \delta(x) \qquad \text{as } \beta \to \infty.$$

Thus we find

$$(\delta(x))\hat{}(\mu) = \frac{1}{\sqrt{2\pi}}.$$

4. From Example 1 we can have similarly

$$\left(e^{-\beta x^2}\right)\hat{} = \sqrt{2\pi} \cdot \frac{1}{\sqrt{4\pi\beta}} e^{-\frac{\mu^2}{4\beta}}.$$

Note the fact

$$G_\beta := \frac{1}{\sqrt{4\pi\beta}} e^{-\frac{\mu^2}{4\beta}} \to \delta(\mu)$$

as $\beta \to 0$. Let $\beta \to 0$, we find

$$1\hat{} = \sqrt{2\pi}\delta(\mu).$$

Thus we have

$$\frac{1}{\sqrt{2\pi}}\hat{} = \delta(\mu).$$

5. From the integral of definition we have formally

$$\delta(x-a)\hat{} = \frac{1}{\sqrt{2\pi}} e^{ia\mu}.$$

We conclude that through approximation, Fourier transform can be defined for many functions that are not in the space $L^1(I\!R^1)$. The functions $f(x) = \delta(x)$ and $f(x) = c$ are examples.

We comment that the coefficient $1/\sqrt{2\pi}$ in the definition of the Fourier transform may be different in other books.